Statistical Analysis of Stochastic Multi-Robot Boundary Coverage

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Abstract—We present a novel analytical approach to computing the population and geometric parameters of a multi-robot system that will provably produce specified boundary coverage statistics. We consider scenarios in which robots with no global position information, communication, or prior environmental data have arrived at uniformly random locations along a simple closed or open boundary. This type of scenario can arise in a variety of multi-robot tasks, including surveillance, collective transport, disaster response, and therapeutic and imaging applications in nanomedicine. We derive the probability that a given point robot configuration is saturated, meaning that all pairs of adjacent robots are no farther apart than a specified distance. This derivation relies on a geometric interpretation of the saturation probability and an application of the Inclusion-Exclusion Principle, and it is easily extended to finite-sized robots. In the process, we obtain formulas for (a) an integral that is in general computationally expensive to compute directly, and (b) the volume of the intersection of a regular simplex with a hypercube. In addition, we use results from order statistics to compute the probability distributions of the robot positions along the boundary and the distances between adjacent robots. We validate our derivations of these probability distributions and the saturation probability using Monte Carlo simulations of scenarios with both point robots and finite-sized robots.

I. INTRODUCTION

Multi-robot systems comprised of large numbers of inexpensive, relatively expendable platforms have the potential to perform tasks on large spatial and temporal scales quickly, robustly, and with little to no human supervision. The production and deployment of such collectives is approaching feasibility due to recent advances in computing, sensing, actuation, power, control, and 3D printing. In the last few years, miniaturization of these technologies has led to many novel platforms for multi-robot applications, including micro aerial vehicles and nano air vehicles [1], [2] for tasks such as exploration, mapping, environmental monitoring, surveillance, and reconnaissance. At even smaller scales, micro-nano systems are currently being developed for micro object manipulation and biomedical applications, including molecular imaging, drug and gene delivery, therapeutics, and diagnostics [3], [4]. It is now possible to design nanoparticles, DNA machines, synthetic bacteria, and magnetic materials that can move, sense, and interact in a controlled fashion, similar to robotic platforms [5]. These nanorobots will need to be deployed in massive numbers; for instance, 10^12 to 10^13 nanoparticles would be required to deliver drugs to a tumor or yield a visible signal for sensing [6].

Many multi-robot applications will involve a boundary coverage task, in which robots must arrange themselves along the boundary of a region or object according to a specified density. Possible applications include cooperative manipulation and transport of payloads, surveillance tasks such as perimeter patrolling, and disaster response tasks such as cordoning off a hazardous area or extinguishing a fire. In nanomedicine, therapeutic and imaging applications will require the use of ligand-coated nanoparticles that can bind selectively to cell surfaces with high receptor densities [7].

This paper addresses boundary coverage tasks in which the robots have extremely limited capabilities, as in micro-nanorobotic applications, or in which it is impractical or impossible to use GPS, communication, or prior environmental data, for instance in disaster response operations and intelligence-surveillance-reconnaissance missions. In addition, the boundary coverage task must be accomplished through stochastic robot behaviors, which arise from noise due to sensor and actuator errors; randomness in robot encounters with the boundary; and, for nanorobots, the effects of Brownian motion and chemical interactions [3]. There is a sizable body of work on designing stochastic robot control policies that produce different types of desired collective behaviors in multi-robot systems, including assembly and self-assembly [8], [9], [10], [11] and task allocation based on spontaneous robot decisions [12], [13], [14], [15]. Encounter-dependent task allocation strategies are most closely related to our stochastic coverage problem, but existing work either deals with scenarios where encountered objects are small (on the scale of the robot) [16], [17] or where large objects are covered dynamically by the robots [18]. In contrast to previous work, we address a static stochastic coverage scenario in which the encountered object or region is large compared to the robots.

This paper presents an analytical framework for computing the robot physical and sensing parameters and the robot population that will provably achieve boundary coverage statistics that may be of interest in multi-robot applications. As stated previously, we assume that the robots have no global position information, no inter-robot communication, and no prior information about the boundary location or geometry. First, we derive the probability that a robot configuration around a boundary is saturated, meaning that each adjacent pair of robots lies within a certain distance. The quantity that this distance signifies depends on the application. For instance, when saturation corresponds to full sensor coverage of the boundary, the distance represents the

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For finite-sized robots of radius $\delta$, we generally choose a value of $d$ that ensures that no robot will fit in between two adjacent robots whose distance is at most $d$. Evidently, the choice $d = 2\delta$ ensures this, and will be used hereafter. No such $d$ naturally suggests itself for point robots, for whom $d$ can be selected arbitrarily. With this in mind, we formulate our problem.

**Problem Statement** Given a configuration in which each of $n$ robots uniformly randomly attach to $\gamma$, compute:

1. the probability $p_{\text{sat}}$ that they saturate the curve,
2. the probability density functions (pdf’s) of each robot on the curve, and
3. the pdf of the inter-robot distance between adjacent robots.

Note that this statement deals with four cases, for the robots can be either point or finite-sized, and the curve can be either closed or open. Our next step will be to save labor by cutting down these four cases to two.

**III. Reduction from Open to Closed Curve**

In this section, we reduce any problem for a robot configuration on an open curve to an equivalent one on a closed curve, thereby simplifying our computations. Suppose first that $\gamma$ is closed. Cut $\gamma$ open at $\gamma(0) = \gamma(1)$, and unwrap it on the positive real axis, placing $\gamma(0)$ at zero. Since $\gamma$ has unit length, it unwraps completely onto the unit interval $I$. Note that unwrapping $\gamma$ leaves distances between robots unchanged. The robot positions $\gamma(t_i')$ now fall on positions $t_i$ on $I$. Observe that $t_i'$ is identified with both endpoints of $I$, as $\gamma$ was a closed curve. We will use this important property repeatedly in the coming discussion.

Now start at $t_i'$ and move rightwards along $I$, labeling robot positions in order until the right endpoint of $I$ is reached, taking us back to $t_r'$. This results in the sequence $(t_1 = t_1', t_2, \ldots, t_n)$, which is a permutation of the unordered points $t_i'$. Since $t_1 = t_1'$ is identified with both ends of $I$, it is considered to fall to the left of $t_2$, as well as to the right of $t_n$, at the same time. Thus, the distance between $t_1$ and $t_2$ uses $t_1 = 0$, and equals $t_2$; on the other hand, the distance between $t_n$ and $t_1$ makes use of $t_n = 1$, and evaluates to $1 - t_n$. To make results simple, we define $t_{n+1} = t_1$, pretending that the index wraps around. Also, use the value $t_{n+1} = 1$ when computing the distance $t_i + t_{n+1}$ for $1 \leq i \leq n$. Saturation requires each inter-robot distance of the form $t_{i+1} - t_i$ to be bounded above by $d$. Next, suppose that $\gamma$ is an open curve, to which $n$ robots attach precisely as before. Unwrap the curve as before onto $I$. Because $\gamma$ is open, we have $t_i' \neq 0$ in general. Consequently, labeling robot positions in order leads to $t_1 \neq 0$ in general. Like in the case of closed curves, saturation implies that $t_{i+1} - t_i \leq d$, for $1 \leq i \leq n - 1$. However, since $\gamma$ is open, saturation forces two additional constraints: $t_1 \leq d$ and $1 - t_n \leq d$.

Introduce an artificial point robot $t_0$, and identify it with both ends of $I$. This robot behaves exactly like $t_1$ in the case of closed $\gamma$. Define $t_{n+1} = t_0$, analogous to closed curves, except that the index wraps back to 0 instead of 1. As before, always take $t_{n+1} = 1$ for computing $t_{n+1} - t_n$. Now
saturation needs $t_1 - t_0 \leq d$, and $t_{n+1} - t_n = 1 - t_n \leq d$, which are equivalent to the two extra constraints. This scenario is identical to having $n+1$ robots on a closed curve. Also, note that the position of $t_1$ on the open curve matches that of $t_{n+1}$ on the closed curve.

An additional observation helps us with the reduction for finite-sized robots. Suppose that the closed curve $\gamma$ having $n$ finite-sized robots is unwrapped on $\mathcal{I}$ as before. The first robot centered at $t'_1 = t_1 = 0$ splits in half when $\gamma$ unwrapped. It spans the endpoints of $\mathcal{I}$ to occupy the disconnected interval $F_1 = [0, \delta] \cup [1 - \delta, 1]$, splitting into half-circles. Every other robot is centered at some $t_i \in \mathcal{I}$, and physically occupies the interval $F_i = [t_i - \delta, t_i + \delta]$ now. Define $t_{n+1} = t_1$ as before, pretending that the index wraps around. Note that the $n$ robots take up a length of $2n\delta$ altogether on $\mathcal{I}$. For saturation, we require that no robot can fit in the space between two consecutive ones. This leads $n$ constraints of the form $t_{i+1} - t_i < 2\delta$.

When $\gamma$ is open, however, $t_1 \neq 0$ in general, leading to two extra constraints, $t_1 < 2\delta$ and $1 - t_n < 2\delta$. As before, we introduce the artificial robot $t_0$; however, it must be placed in a way that resembles that of $t_1$ for closed curves, yet does not take away any space from $\mathcal{I}$ for other robots to occupy. To meet both these requirements, we place $t_0$ outside $\mathcal{I}$, splitting it into half-circles so that $F_0 = [-\delta, 0] \cup [1, 1 + \delta]$. Also, we have $t_{n+1} = 1$, coinciding with the right endpoint of $\mathcal{I}$. For saturation, $t_1$ can be at most $\delta$ away from $t_0$, and likewise $t_n$ can be at most $\delta$ away from $t_{n+1}$, taking care of the two extra constraints automatically. This is equivalent to having $n + 1$ robots on a closed curve of length $1 + 2\delta$.

In short, for point robots, a problem with parameters $(n, d)$ on the open curve is equivalent to one with parameters $(n + 1, d)$ on the closed curve. For finite-sized robots, a problem with parameters $(n, \delta)$ on the open curve of unit length is equivalent to one with parameters $(n + 1, \delta)$ on a closed curve of length $1 + 2\delta$. Due to these reductions, the future sections will compute results mainly for closed curves, commenting on the open curve case only when needed.

### IV. A BRUTE FORCE APPROACH TO COMPUTING $p_{\text{sat}}$

We now turn back to $n$ point robots on a closed curve. We first characterize $p_{\text{sat}}$ as the ratio of the measure of $\mathcal{E}$, the space of saturated robot configurations, to the measure of $\Omega$, the sample space of all possible robot configurations. Since $t_1 = 0$, the sample space $\Omega$ consists of all points $(t_2, t_3, \ldots, t_n)$ with coordinates obeying $0 \leq t_i \leq t_{i+1} \leq 1$. This space $\Omega$ forms a simplex in $\mathbb{R}^{n-1}$ with vertices $v_1 = (0, 0, \ldots, 0, 0), v_2 = (0, 0, \ldots, 0, 1), v_3 = (0, 0, \ldots, 0, 1, 1), \ldots, v_n = (1, 1, \ldots, 1, 1)$. We call $\Omega$ the event simplex. The measure of $\Omega$ is its $(n-1)$-dimensional volume,

$$|\Omega| = \int_0^1 \int_{t_2}^1 \int_{t_3}^1 \cdots \int_{t_{n-1}}^1 dt_n \cdots dt_3 dt_2 = \frac{1}{(n-1)!}$$

We now characterize the space $\mathcal{E}$ of saturated robot configurations. To do this, we need to identify the set of robot position ranges, or saturating intervals $I_i$, $i = 2, \ldots, n$, that guarantee a saturated robot configuration. We derive the $I_i$ from the intersection of the left saturating interval $L_i$ and right saturating interval $R_i$ for each robot position $t_i$. These two intervals denote the range of positions that result in the saturating conditions $(t_i - t_{i-1}) \in [0, d]$ and $(t_{i+1} - t_i) \in [0, d]$, respectively.

To ensure the condition $(t_i - t_{i-1}) \in [0, d]$, we define the left saturating interval for each robot position $t_i$ as

$$L_i = [t_{i-1}, \min(t_{i-1} + d, 1)], \quad i = 2, 3, \ldots, n$$

To ensure the condition $(t_i - t_{i-1}) \in [0, d]$, we define the right saturating interval for position $t_n$ as

$$R_n = \max(0, 1 - d)$$

We obtain the right saturating intervals $R_i$ for the remaining positions $t_i$ using the following approach. We will refer to a sequence of robot positions as $d$-separated if the distance between any two consecutive robot positions is precisely $d$. Consider a particular $t_2 \in I_2$, and define a sequence of $d$-separated positions $t_3, t_4, \ldots, t_n$. If the resulting $t_n \in R_n$, the choice of $t_2$ is sufficient to ensure saturation. The leftmost possible positions $t_i$ that will still result in saturation are $t_n = 1 - d, t_{n-1} = 1 - 2d, \ldots, t_2 = 1 - (n-1)d$. Hence, to ensure saturation, we define $R_2 = [\min(1 - d(n-1), 0), 1]$. Extending this reasoning to the remaining robot positions, the following $R_i$ ensure the condition $(t_i - t_{i-1}) \in [0, d]$:

$$R_i = [\min(1 - (n-i+1)d, 1), i = 2, 3, \ldots, n-1$$

Finally, from the intervals in Equation (2), Equation (3), and Equation (4), we obtain the following intervals for the robot positions that lead to saturated configurations:

$$I_i = L_i \cap R_i = [\max(t_{i-1}, 1 - (n-i+1)d), \min(t_{i-1} + d, 1)], \quad i = 2, 3, \ldots, n$$

A point in $\mathcal{E}$ has $t_i \in I_i$ for $i \in \{2, 3, \ldots, n\}$. We can compute the measure of $\mathcal{E}$ as

$$|\mathcal{E}| = \int_{t_2 \in I_2} \int_{t_3 \in I_3} \cdots \int_{t_n \in I_n} dt_n \cdots dt_2$$

Note that if any of the intervals $I_i$ is empty, $\mathcal{E}$ becomes the empty set and the probability of saturation is zero.

This integral can be evaluated on a case-by-case basis, but a naive expansion of the min and max functions in the limits of each $I_i$ could result in an exponential number of subintervals, which is computationally expensive. Thus, the probability of saturation $p_{\text{sat}} = |\mathcal{E}|/|\Omega|$ needs to be determined by other means, for which we explore the geometric approach described in the next section.

### V. GEOMETRIC INTERPRETATION OF $p_{\text{sat}}$

We use an approach to computing $p_{\text{sat}}$ that can be described geometrically and is based on the characterization of the inter-robot distances, which we call the slacks. We will initially consider configurations of point robots. Given a sequence of robot positions $t_i, i = 1, 2, \ldots, n$, define the $i^{th}$ slack to be the distance between $t_i$ and $t_{i+1}$:

$$s_i = t_{i+1} - t_i, \quad i = 1, 2, \ldots, n$$
Recall from section III that since $t_{n+1} = 1$, we have $s_n = 1 - t_n$. Define the slack set $S_n \subset \mathbb{R}^n$ to be the set of all possible slack points $s = (s_1, s_2, \ldots, s_n)$, each of whose components is a valid slack. To characterize this set, we note that valid slacks are always nonnegative and that the sum of the slacks, called the total slack $s$, is always equal to the curve length $l$. Thus, the slack set is defined as

$$S_n = \{ s \in \mathbb{R}^n | s_i \geq 0, \ i = 1, 2, \ldots, n, \sum_{i=1}^n s_i = s \}$$  \hfill (9)

The slack set $S_n$ forms an $(n-1)$-dimensional simplex that we call the slack simplex. The $n$ vertices of $S_n$ are located at $(s, 0, 0, \ldots, 0), (0, s, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, s)$. Hence, $S_n$ is a regular simplex with side length $s\sqrt{2}$. The volume of an $n$-dimensional regular simplex of side $a$ is

$$V_n(a) = \left( \frac{a}{\sqrt{2}} \right)^n \frac{\sqrt{n+1}}{n!}$$  \hfill (10)

This gives the following volume for $S_n$:

$$|S_n| = V_{n-1}(s\sqrt{2}) = \frac{s^{n-1}\sqrt{n}}{(n-1)!}$$  \hfill (11)

Now we describe the set of points $s \in \mathbb{R}^n$ that corresponds to configurations that are saturated, but that do not necessarily conserve the total slack (and thus form a valid robot configuration). For such configurations, $s_i \in [0, d]$ for all $i$. Thus, the set of all points leading to saturation forms an $n$-dimensional hypercube $H_n$:

$$H_n = \{ s \in \mathbb{R}^n | 0 \leq s_i \leq d, \ i = 1, 2, \ldots, n \}$$  \hfill (12)

The set of valid robot configurations that are saturated is therefore the $(n-1)$-dimensional set $E_n \equiv S_n \cap H_n$. The probability of saturation $p_{sat}$ is the volume of this set divided by the volume of the slack simplex, the set of all valid robot configurations. Determining the volume of $E_n$ is not straightforward, but it can be computed using the following approach. Suppose that $A_k \subseteq \{1, 2, \ldots, n\}$ is a $k$-element subset of the slack indices. Define for any $A_k$ the set $E'_n(A_k)$ consisting of all slack points whose components $s_i$, $i \in A_k$, are at least $d$:

$$E'_n(A_k) = \{ s \in S_n | s_i \geq d \ \forall i \in A_k \}$$  \hfill (13)

Note that there is no constraint on the remaining slack coordinates, which may exceed or fall below $d$ as long as $s \in S_n$. For each slack $s_i$, $i \in A_k$, define a reduced slack $s'_i = s_i - d$. We can equivalently define $E'_n(A_k)$ in terms of the reduced slacks as:

$$E'_n(A_k) = \{ s \in S_n | \sum_{i \in A_k} s'_i + \sum_{i \notin A_k} s_i = s - kd \}$$  \hfill (14)

This definition makes it evident that $E'_n(A_k)$ is a regular simplex that conserves the total reduced slack $s - kd$. By Equation (10), its $(n-1)$-dimensional volume is given by

$$|E'_n(A_k)| = V_{n-1}((s-kd)\sqrt{2}) = \frac{(s-kd)^{n-1}\sqrt{n}}{(n-1)!}$$  \hfill (15)

Equation (15) makes sense only if $(s-kd) \geq 0$; equivalently, we need $1 \leq k \leq K$ where $K = |s|/d$. If $k > K$, then $|E'_n(A_k)| = 0$, as there is no subset of $S_n$ in which $s_i > d$, $i \in A_k$. We will next use Equation (15) to compute the volume of $E'_n = S_n \setminus E_n$. This set is defined as

$$E'_n = \{ s \in S_n | \exists s_i \geq d, \ i \in \{1, 2, \ldots, n\} \} = \bigcup_{A_k} E'_n(A_k)$$  \hfill (16)

Note that the union runs over all possible choices of $A_k$. To compute the volume of the union, we use the Inclusion-Exclusion Principle [20] which states that

$$\left| \bigcup_{A_k} E'_n(A_k) \right| = \sum_{k=1}^n (-1)^{k-1} \sum_{k=1}^n |E'_n(A_k)|$$  \hfill (17)

To evaluate the second sum on the righthand side of Equation (17), note that there are precisely $\binom{n}{k}$ subsets of $\{1, 2, \ldots, n\}$ of cardinality $k$, and the volume of each corresponding $E'_n(A_k)$ is given by Equation (15). Also, we need to take this sum only on the range $1 \leq k \leq K$, for otherwise $|E'_n(A_k)| = 0$. From these observations, we obtain

$$|E'_n| = \left| \bigcup_{A_k} E'_n(A_k) \right| = \sum_{k=1}^K (-1)^{k-1} \sum_{k=1}^n |E'_n(A_k)|$$  \hfill (18)

Since $E_n = S_n \setminus E'_n$, $|E_n| = |S_n| - |E'_n|$. Using Equation (11) and Equation (18), we obtain the probability of saturation:

$$p_{sat} = \frac{|E_n|}{|S_n|} = 1 - \sum_{k=1}^K (-1)^{k-1} \binom{n}{k} \left( \frac{1 - kd}{s} \right)^{n-1}$$  \hfill (19)

Finally, from section III, this result trivially extends to open curves by mapping $n \rightarrow n+1$.

A. Evaluating the Intractable Integral in Equation (6)

An immediate consequence of Equation (19) is that we can now evaluate the integral in Equation (6) as $\xi' = p_{sat}|\Omega| = p_{sat}/(n-1)!$. Let $\Sigma$ denote the sum on the right side of Equation (19). Given $n$, $d$, and $s = 1$, this sum $\Sigma$ Equation (19) takes $K$ steps in all to evaluate in a loop.

For efficiency, the $k$-th step computes $\binom{n}{k}$ recursively from $\binom{k-1}{k}$ in $O(1)$ time and the power term $(1 - kd/s)^{n-1}$ by binary exponentiation in $O(\log n)$ time. Note that both $\binom{n}{k}$ and the power term have a size that is bounded above by a polynomial function of the number of input bits. As such, $p_{sat}$ can be computed in polynomial time, with $O(K\log n)$ multiplications in all.

B. Extension to Finite-Sized Robots

We avoided addressing finite-sized robots in section IV, since this case is even more complicated than that of point robots. However, the geometric approach has an easy extension to finite-sized robots. Following section II, define the slack $s_i$ by

$$s_i = t_{i+1} - t_i - 2\delta, \ i = 1, 2, \ldots, n$$  \hfill (20)

where as usual $t_{n+1} = 1$. Given $t_i$ and $s_i$, the next robot is located at $t_{i+1} = t_i + s_i + 2\delta$ and occupies the interval $F_{i+1} = [t_i + s_i + \delta, t_i + s_i + 3\delta]$. 

No matter how the robots are positioned, the total length of the curve not occupied by robots is $1 - 2\delta n$. In addition, in saturated robot configurations, all slacks $s_i$ are too small to fit another robot between the robots at $t_i$ and $t_{i+1}$; that is, $s_i \in [0, 2\delta]$. Hence, we have the following values for the total slack $s$ and the saturating distance $d$:

$$s = 1 - 2\delta n \quad (21)$$
$$d = 2\delta \quad (22)$$

Using the same reasoning as in the point robot case, the set of possible slack points for finite-sized robot configurations consists of the slack simplex Equation (9) with $s$ defined in Equation (21), and the set of all points corresponding to saturated configurations forms the hypercube Equation (12) with $d$ defined in Equation (22). Repeating the analysis developed for point robots, we find that the formula for $p_{sat}$ is given by Equation (19) with the newly defined values of $s$ and $d$.

Finally, we comment briefly about the analogous results for open curves. The reduction in section III from open to saturated configurations forms the hypercube Equation (12) in Equation (21), and the set of all points corresponding to these configurations is the event simplex $\Omega$. Since the order statistics $t_i$ are just a permutation of their parents, their joint pdf $f(t_1,t_2,\ldots,t_n)$ is identical to that of their parents. Note that $t_1$ is fixed and thus has a degenerate marginal pdf, and we omit it from further consideration. We can obtain the pdf of other order statistic $t_i : i = 2, 3, \ldots, n$, by repeatedly marginalizing over the remaining $n-2$ statistics. We do so in two steps. In the first step, notice that the order statistics $t_j : j = i+1, i+2, \ldots, n$ to the right of $t_i$ each need to be integrated over the interval $R_j = [t_{j-1}, 1]$. This leads us to a marginal density of the form $f(t_2,\ldots,t_i)$, consisting only of $t_i$ and the statistics to its left. Next, we marginalize away the statistics $t_k : k = 2, \ldots, i-1$ to the left of the desired one by integrating each of them over the intervals $L_k = [0,t_i]$. This gives the result

$$f(t_i) = \int_{L_k} \int_{R_j} f dt_n \cdots dt_{i+1} dt_{i-1} \cdots dt_2 = \binom{n}{i} (1-t_i)^{n-i} \quad (26)$$

where we wrote

$$\int_{L_k} \int_{R_j} \int_{t_{i+1} \in R_{i+1}} \cdots \int_{t_n \in R_n}$$

and $f$ for the joint density of the order statistics.

Define the Beta density [22, pp.42-43] Beta$(t|a,b)$ by

$$\text{Beta}(t|a,b) := \frac{1}{B(a,b)} t^{a-1} (1-t)^{b-1} \chi_{\mathcal{P}} \quad (27)$$

This density has mean and variance given by

$$E[\text{Beta}(t|a,b)] = \frac{a}{a+b}$$
$$\text{Var}[\text{Beta}(t|a,b)] = \frac{ab}{(a+b+1)(a+b)^2}$$

Then it is easy to see that each order statistic is a Beta variate of the form

$$f(t_i) = \text{Beta}(t_i|i,n-i+1) \quad (28)$$

with mean and variance given by

$$E[t_i] = \frac{i-1}{n} \quad (29)$$
$$\text{Var}[t_i] = \frac{(i-1)(n-i+1)}{n^2(n+1)} \quad (30)$$

Intuitively, Equation (28) indicates that $t_i$ is the $(i-1)^{st}$ variable from among the $n-1$ picked on $\mathcal{P}$. The $n-1$ order statistics $t_2$ through $t_n$ divide $\mathcal{P}$ into $n$ subintervals. On average, these subintervals have equal length, placing the expected location of $t_i$ at $\frac{i-1}{n}$.

We next compute the pdf’s of the slacks. Note that any particular slack, say the last slack $s_n$, can be determined as a function of the remaining ones, as $s_n = 1 - \sum_{i=1}^{n-1} s_i$. To define the joint pdf of the slacks, keep this slack apart, and define the domain

$$D := \{(s_1,s_2,\ldots,s_{n-1}) : 0 \leq s_i \leq 1, 0 \leq \sum_{i=1}^{n-1} s_i \leq 1 \} \quad (31)$$
This domain is the interior of an \((n-1)\) dimensional simplex embedded in \(\mathbb{R}^{n-1}\) and is easily seen to have the \((n-1)\) dimensional volume \(|D| = \frac{1}{(n-1)!}\) by a nested integral identical to Equation (1), leading to the joint pdf

\[
f(s_1, s_2, \ldots, s_{n-1}) = (n-1)!D
\] (32)

Unlike the order statistics which need \(t_i \leq t_{i+1}\), the slacks have no such restriction. Since any reordering of slacks makes no difference to Equation (32), every slack has the same marginal pdf by symmetry. This includes the slack \(s_n\) that was omitted from the joint pdf, for we could have set up Equation (25) by setting apart another slack, say \(s_{n-1}\) instead. Thus we need to determine only one marginal pdf, say \(f(s_1)\).

To do so, we follow a marginalization process similar to that for the order statistics. Note that every slack \(s_j: j = 2, \ldots, n-1\) to the right of \(s_1\) lies on the interval \(R_j = [0, 1 - \sum_{i=1}^{j} \delta s_j]\). Integrating the joint slack pdf in Equation (32) over these intervals leads us to

\[
f(s_i) = \text{Beta}(s_i|1, n-1), i = 1, 2, \ldots, n
\] (33)

We add a note of caution here that will become relevant to the case of finite-sized robots. Observe that since \(t_i = \sum_{j=1}^{i-1} \delta s_j\), it may be tempting to compute the marginal density \(f(t_i)\) by performing \(i-1\) nested convolutions of the slack pdf’s. However, it can be seen while computing the marginal pdf’s that slacks are mutually dependent, so that for example \(f(s_1, s_2) \neq f(s_1)f(s_2)\). Deriving \(f(s_i)\) by repeated integration sidesteps this issue of dependency.

Finally, all results from Equation (25) through Equation (33) extend trivially to open curves as follows. Since the reduction process from open to closed curves increments the number of robots, the joint pdf for \(n\) robots on an open curve is

\[
f(t_1, t_2, \ldots, t_n) = n!1_\Omega
\] (34)

where \(\Omega\) is now an \(n\)-dimensional simplex embedded in \(\mathbb{R}^n\).

The statistics \(t_i\) for a closed curve resembles that of \(t_{i+1}\) for closed curves, giving us

\[
f(t_i) = \text{Beta}(t_i|i, n-i+1), i = 1, 2, \ldots, n
\] (35)

\[
f(s_i) = \text{Beta}(t_i|1, n), i = 1, 2, 3, \ldots, n+1
\] (36)

**B. Extension to Finite-Sized Robots**

In the case of finite-sized robots, the slacks \(s_i\) are the order statistics of points uniformly randomly chosen on the interval \([0, s = 1 - 2\delta n]\). Had \(s\) been unity, the pdf of each \(s_i\) would have been given by the Beta pdf in Equation (33). If we divide each \(s_i\) by \(s\), the scaled random variables \(s_i/s\) will be distributed over \(\mathcal{X}\) according to this Beta pdf. From this, we derive the following expressions for the pdf of \(s_i\):

\[
f(s_i) = s \cdot \text{Beta}(t_i|1, n-1)
\] (37)

However, determining the pdf of \(t_i\) does not readily admit an analytic form. Having set \(t_1 = 0\), from Equation (20) we know that

\[
t_2 - 2\delta = s_1
\] (38)

Consequently, \(t_2 - 2\delta\) will have the same pdf as \(s_1\), or

\[
f(t_2) = s \cdot \text{Beta}(t_1|1, n-1) + 2\delta
\] (39)

The case of \(t_3\) is different. From Equation (20),

\[
t_3 - 4\delta = s_1 + s_2
\] (40)

Determining \(f(t_3)\) by convolution fails due to the fact that slacks are dependent, as the note of caution makes clear in section VI-A. Unlike with point robots, we do not have a general expression for the joint pdf of finite-sized ones. Thus, the marginalization procedure of section VI-A fails, and we have no analytic expressions for the pdf of \(t_3\) through \(t_n\).

**VII. Simulations**

We implemented Monte Carlo simulations in which various numbers \(n\) of either point robots or finite-sized robots are stochastically allocated to the boundary of a unit circle. In the point robot simulations, \(n\) points were chosen uniformly randomly from \(\mathcal{X}\) and sorted in order. The slacks were computed, and those robot configurations that met the saturation criterion were counted as favorable events. The finite-sized robots implemented the following algorithm:

**Algorithm SLACK-ATTACH\((n, \delta)\)**

1) Set \(t_1 \leftarrow 0\).
2) Pick an array \(arr[1..n-1]\) of uniformly randomly distributed points on the interval \([0, s]\).
3) Sort \(arr\) in increasing order.
4) Compute the individual slacks as \(s_1 \leftarrow arr[1], s_i \leftarrow arr[i+1] - arr[i]\) for \(2 \leq i \leq n-1\), and \(s_n \leftarrow s - \sum_{1 \leq i \leq n-1} s_i\).
5) Compute \(t_{i+1} = t_i + s_i + 2\delta\) for \(1 \leq i \leq n-1\).

Robot configurations that satisfied \(s_i = d \leq 2\delta\) were counted as favorable events.

Figure 2 compares three-dimensional plots of the theoretical and Monte Carlo-averaged (over 20000 trials) \(p_{sat}\) varying with \(n\) and \(d\) for point and finite-sized robots. For both robot types, the close match between the theoretical and Monte Carlo-averaged plots validate our formula for \(p_{sat}\). Table I and Table II show a quantitative comparison of \(p_{sat}\) computed from Equation (19) and \(p_{sat}\) averaged over 20000 Monte Carlo trials for different combinations of \(n\) and \(d\). Table III and Table IV display the corresponding results for finite-sized robots. For both robot types, the theoretical predictions of \(p_{sat}\) closely match the Monte Carlo simulation averages.

We also plotted the frequency of 5000 samples of a particular robot position and a particular slack from Monte Carlo trials with \(n = 5\) point robots. As Figure 3 shows, both frequency plots can be fit to the Beta densities predicted by Equation (28) and Equation (37).
Theoretical Monte Carlo

Fig. 2. Plots of theoretical (left column) and Monte Carlo-generated (right column) $p_{sat}(n,d)$ values for point (upper row) and finite-sized (lower row) robots.

**TABLE I**

<table>
<thead>
<tr>
<th>$d$</th>
<th>Theoretical $p_{sat}$</th>
<th>Monte Carlo $p_{sat}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.20</td>
<td>0.4929</td>
<td>0.4925</td>
</tr>
<tr>
<td>0.25</td>
<td>0.7898</td>
<td>0.7913</td>
</tr>
<tr>
<td>0.33</td>
<td>0.9653</td>
<td>0.9651</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9995</td>
<td>0.9997</td>
</tr>
</tbody>
</table>

**TABLE II**

<table>
<thead>
<tr>
<th>$n$</th>
<th>Theoretical $p_{sat}$</th>
<th>Monte Carlo $p_{sat}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.2500</td>
<td>0.2467</td>
</tr>
<tr>
<td>4</td>
<td>0.5000</td>
<td>0.5029</td>
</tr>
<tr>
<td>5</td>
<td>0.6875</td>
<td>0.6885</td>
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<tr>
<td>6</td>
<td>0.8125</td>
<td>0.8124</td>
</tr>
<tr>
<td>7</td>
<td>0.8996</td>
<td>0.8999</td>
</tr>
<tr>
<td>8</td>
<td>0.9375</td>
<td>0.9383</td>
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**TABLE III**

<table>
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<tr>
<td>0.05</td>
<td>0.0000</td>
<td>0.0000</td>
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<td>0.06</td>
<td>0.0029</td>
<td>0.0029</td>
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<tr>
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</table>

**TABLE IV**

<table>
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<tr>
<th>$n$</th>
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<th>Monte Carlo $p_{sat}$</th>
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<tr>
<td>5</td>
<td>0.0000</td>
<td>0.0000</td>
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<tr>
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<td>0.0254</td>
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<td>0.4221</td>
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<td>0.9401</td>
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**VIII. CONCLUSIONS AND FUTURE WORK**

We have demonstrated approaches to the statistical analysis of quantities that are associated with stochastic coverage of a simple boundary by a set of uniformly randomly distributed robots. We developed a geometric interpretation of the probability that a given robot configuration is saturated and derived this probability using the Inclusion-Exclusion Principle. We also used results from order statistics to determine the probability distributions of the robot positions along the boundary and the distances between adjacent robots. We validated our derived formulas for these probability distributions and the saturation probability using Monte Carlo simulations of a large number of scenarios. The work presented here can be extended in several directions:

- **Stochastic coverage by asynchronously attaching robots**: Many experimental scenarios will involve asynchronous attachment of robots to a boundary. In these cases, each attaching robot searches for an available attachment location on the boundary and picks one at random. One possible algorithm for simulating this process with finite-sized robots is as follows:

**Algorithm** `ASYNC-ATTACH(n, \delta)`

1) Set $t'_1 \leftarrow 0$
2) For each $i : 2 \leq i \leq n$ do
   a) Find the list of intervals in which the center $t'_i$ of the next robot can be placed.
   b) Choose one such interval at random and one point $p$ at random from this interval.

Fig. 3. Frequency plots of 5000 samples of $t_2$ and $s_2 = s_3 - t_2$ from Monte Carlo trials for point robots. Both plots are fit to a Beta($\theta | 1, 4$) density function.
c) Set $t'_i \leftarrow p$.

3) Sort $t'_i$ in increasing order to get $t_i$.

The parent variables $t'_i$ assigned by ASYNC-ATTACH are not i.i.d. since each $t'_i$ is a function of $t'_1, t'_2, \ldots, t'_{i-1}$. This makes the determination of their order statistics a much more challenging task [21, Ch.5] for which a complete analytical solution may be impossible. Hence, an important future step would be to extract useful qualitative information about the distributions generated by ASYNC-ATTACH and other asynchronous attachment algorithms. Also pertinent would be to devise asynchronous attachment algorithms that lend themselves to tractable analysis.

b) Probability distributions induced by saturation: As an extension of the material in Section VI, we plan to derive the pdf’s of $t_i$ and $s_i$ given that the robot configuration is saturated.

c) Comparison with deterministic algorithms: Multi-robot planar boundary coverage is an NP-Hard problem [23]. Deterministic algorithms for this problem generally employ heuristics to reduce running time. Our approach is essentially a randomized algorithm to address the same intractability. Deterministic algorithms provide more guarantees on runtime and correctness than randomized algorithms, while the latter may be simpler to implement [24]. Our future work will investigate scenarios that favor one approach over the other; for example, we would wish to determine the better coverage approach when collisions with obstacles or between robots need to be avoided.

d) Applications to multi-robot transport problems: We are currently investigating the problem of developing strategies for multi-robot collective transport that are robust to payload type, environment layout, and transport team size, much like group retrieval strategies employed by certain species of ants [25]. We will consider scenarios in which robots stochastically allocate themselves around a payload and then proceed to transport the payload as a team. We will apply our statistical analysis results to characterize dynamical properties of the robot-load system during transport given an initial stochastic allocation. For instance, we can investigate the probability that robots arranged in a random configuration around a certain payload will be able to successfully lift it off the ground.

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REFERENCES


