N robots moving in continuous state space \( y \in \mathbb{R}^d \), \( d \in \{2, 3\} \).

At any given time, a robot's actions are determined by one of a set of controllers.

- A controller causes a robot to fulfill a task or subtask.

An Analogue to Chemical Reaction Network (CRN):

\( X_i = \) species \( i = \) robot that is performing task \( i \), object of type \( i \), or robot-object complex of type \( i \), \( i \in \{1, 2, \ldots, S\} \).

Example reaction:

\[ X_m + X_n \rightarrow X_p \]

\( k_{ij} = \) reaction rate constant

\( \text{complex } i \rightarrow \text{complex } j \)

To be able to abstract the microscopic model (CRN model of robot task transitions + set of controllers that accomplish the tasks) to the macroscopic model (set of differential equations), the task transitions must be executed in a way that conforms to the fundamental hypothesis of the stochastic formulation of chemical kinetics:
- A reaction that converts complex i into complex j is characterized by a stochastic reaction constant \( c_{ij} \) defined such that:

\[
c_{ij} \Delta t = \text{average probability that a particular combination of elements of the species types in complex i will transition to the species types in complex j in the next infinitesimal time interval} \Delta t.
\]

- \( k_{ij} \) is proportional to \( c_{ij} \).

- Reaction propensity \( a_{ij} \):

\[
a_{ij} \Delta t = \text{probability that the reaction that converts complex i into complex j will occur in the next } \Delta t.
\]

- \( h_{ij} \) = number of distinct combinations of elements that can undergo the reaction.

- \( a_{ij} = c_{ij} \cdot h_{ij} \)

I. Interaction-Dependent Switching \( X_m + X_n \xrightarrow{k_{ij}} \text{products} \)

- It is rare that 3 or more elements encounter each other simultaneously. Hence we will only consider bimolecular reactions in which the reactants consist of species m and n. [Note, however, that reactions with \( \geq 3 \) reactants can approximate a sequence of 2-reactant encounters that occur very fast.
- Robots move in a bounded domain with volume $V$ (area $A$ in 2 dimensions).

- For the fundamental hypothesis to be true, the system must be "well-mixed."

How can this be implemented/checked in robotic systems?

- Can specify that robots perform a random walk with fixed speed and verify that they are uniformly randomly distributed over the domain.

- Can check that diffusion rate associated with the random walk is much higher than the reaction rates.

[In molecular CRN's, "well-mixed" means that the species move randomly & quickly, giving every possible reactant set an equal chance to have a collision.]

$$C_{ij}^{\Delta t} = C_{ij} \cdot C_{ij}^{\Delta t}$$

$$C_{ij}^{\Delta t} = \text{prob. that a random pair of reactant elements of complex } i \text{ will encounter each other in the next time interval } \Delta t$$

$$C_{ij} = \text{prob. that these reactants will form complex } j \text{ given that they are in close proximity. This is a parameter that we can design for macroscopic sys. objectives.}$$
- To satisfy the fundamental hyp, $c_{ij}$ and $c_{ij}$ must both be independent of $St$.

- $N_i =$ number of elements of species $i$

  $c_{ij} = \frac{k_{ij}}{V}$, $h_{ij} = N_m N_n$ if $m \neq n$

  $c_{ij} = 2k_{ij}$, $h_{ij} = N_m (N_m - 1) / 2$ if $m = n$

Robot controller:

Robot in complex $i$ that encounters another robot/element with which it can "react" to form complex $j$ computes a uniformly distributed random number $u \in [0, 1]$ and follows through with the transition if $u \leq c_{ij}$.

II. Spontaneous Task Switching $X_i \xrightarrow{k_{ij}}$ products

- Unimolecular reactions - reactant consists of one species

- To satisfy the fund. hyp., a robot at task $i$ performs the transition at probability per unit time $c_{ij}$.

  $c_{ij} = k_{ij}$, $h_{ij} = N_i$

- $k_{ij}$ is also called a "transition rate" for these types of reactions: $X_i \xrightarrow{k_{ij}} X_j$
Robot Controller:

Robot doing task i computes a uniformly random number $u \in [0, 1]$ at each (very small) timestep $\Delta t$ and executes the transition if $u < k_{ij} \Delta t$.

The number of transitions governed by $k_{ij}$ that occur in $\Delta t$ has a Poisson distribution with parameter $k_{ij} \Delta t$. 