Consensus Problems in Multi-Robot Systems

Examples

1. Rendezvous: robots must know the rendezvous point
2. Information consensus: robots must have a consistent view of information state
   Ex: Center of shape of a formation, rendezvous time, length of perimeter being monitored, direction of motion, target location of payload
3. Formation control, 3b. flocking
4. Attitude alignment (also called agreement protocols)

- Consensus algorithms are distributed: only neighbor-to-neighbor interactions
- Robots update the value of the state of interest based on their neighbors' values
- Goal: Design an update law so that the states of all robots converge to a common value.

- Analysis framework:
  - Based on tools from matrix theory, algebraic graph theory, and control theory.
  - Spectral and structural properties of networks (eigenvalues) speed of information diffusion in consensus algorithms
(Communication)
- Comm. network allows continuous comm. / Comm. bandwidth is large ⇒ Info state update of each vehicle modeled as ODE
- Comm. data arrive in discrete packets ⇒ modeled as difference equation
- Team's comm. topology rep. by a directed graph (digraph).
  - Comm. dropouts may occur, vehicle motion away from other ⇒ time-varying topology
- Most common continuous consensus alg:

\[ \dot{x}_i(t) = -\sum_{j=1}^{n} a_{ij}(t) (x_i(t) - x_j(t)) \quad i=1, \ldots, n \]

\( n \) = # of vehicles (robots)
\( x_i(t) \) = info state of \( i \)th vehicle (robot) at time \( t \)
\( a_{ij}(t) \) = \( (i,j) \) entry of adjacency matrix of the comm. graph at time \( t \)

Vertices: \( V = \{1, \ldots, n\} \) (also called nodes)
Edges: \( E = \{(i,j) \mid \text{robot } j \text{ can obtain info from robot } i\} \)
- Directed \( G \) = ordered pair if graph is directed
- Undirected \( G \) = \( E = \{(i,j) \mid \text{robots } i \text{ and } j \text{ can obtain square info from each other}\} \)
- The digraph of a matrix \( M \) is the digraph with vertex set \( V = \{1, \ldots, n\} \) such that \( \exists \) an edge from \( j \) to \( i \) iff \( M_{ij} \neq 0 \).
Adjacency Matrix: \( A \in \mathbb{R}^{n \times n} \)

of a digraph

\( a_{ij} \) is a positive weight if \((j, i) \in E\)
\( a_{ij} = 0 \) if \((j, i) \notin E\)

If the weights aren't relevant, then \( a_{ij} = 1 \) for all \((j, i) \in E\).

Self-edges \((a_{ii} > 0)\) are allowed.

\( A \) is symmetric for undirected graphs.

Laplacian Matrix: \( L \in \mathbb{R}^{n \times n} \)

of a digraph

\[ l_{ii} = \sum_{j \neq i} a_{ij} \]
\[ l_{ij} = -a_{ij}, \; i \neq j \]

If \((j, i) \in E\), then \( l_{ij} = -a_{ij} = 0 \)

\( L \) is symmetric for an undirected graph.

Properties:

1. \( l_{ij} \leq 0, \; i \neq j \)
2. \( \sum_{j=1}^{n} l_{ij} = 0, \; i = 1, 2, \ldots, n \) (rows sum to 0)

Undirected \( G \): \( L \) is positive semidefinite; all nonzero \( \lambda_i \)'s of \( L \) are positive.

Directed \( G \): all nonzero \( \lambda_i \)'s of \( L \) have positive real part.

Undirected Graph:

\[ \lambda_i(L) = \text{ith smallest} \; \lambda_i \; \text{of} \; L, \; \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \]

Quantifies the algebraic connectivity and the convergence rate of consensus algorithms, and is positive if \( G \) is connected (i.e., an undirected path exists between every pair of distinct nodes).
From the model for $x_i(t)$, $x_i(t) \to$ inf of state of the neighbors of $i$.

- Ensures that all $x_i(t) \to \bar{x}$ (common value)
  But doesn't dictate a specific value.

- In general, the common value is a convex combination of the $x_i(0)$.

Can write (1) as: $\dot{x}(t) = -L(t)x(t)$

Consensus is achieved if $\forall x_i(0)$ and all $i, j = 1, ..., n$,
$|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$.

Convergence Analysis of Consensus Alg w/ Time-Invariant Communication Topologies

$\lambda$ is constant in this case. $\lambda$ = eigenvalue

- Recall: $0$ is an $\lambda$ of $-L$, all nonzero $\lambda$'s of $-L$
  have neg. real parts.

  $L1 = 0 \Rightarrow \text{span } \{1\} \subseteq \text{kernel of } L$

  $\Rightarrow$ If $0$ is a simple $\lambda$ of $L$, then $x(t) \to \bar{x}1$.

  $\Rightarrow \quad |x_i(t) - x_j(t)| \to 0$ as $t \to \infty \quad \forall \quad i, j = 1, ..., n$

- If the digraph of $L$ is strongly connected, then $L$
  $0$ is a simple $\lambda$ of $L$.

  But not necessarily vice versa. (This is not a necessary condition)
0 is a simple eigenvalue of $\preceq$ iff the associated digraph of $\preceq$ contains a rooted directed spanning tree (rdst).

Iff the graph has at least one vertex with a directed path to all the other vertices.

Undirected graph: This condition is equivalent to being connected.

This condition is necessary and sufficient for model $\dagger$ to achieve consensus.
Equilibrium State of Consensus Algorithm

Assume $a_{ij}$ constant, network topology is fixed.

If the digraph contains a rooted directed spanning tree, then
\[
\lim_{t \to \infty} e^{-\frac{1}{\tau}t} \to \pm V^T, \quad V \in \mathbb{R}^{n \times 1}, \quad \sum_{j=1}^{n} v_j = 1, \quad v_j \geq 0,
\]

\[
\Rightarrow x_i(t) \to \sum_{j=1}^{n} v_j x_j(0) \text{ as } t \to \infty.
\]

If some $v_j = 0$, then the info states $x_j$ don't contribute to the equilibrium.

Define $M = \max_i L_{ii} \geq \frac{1}{\tau}$. $M$ is nonnegative.

Diagonal entries of $L$

$V$ is a nonnegative left eigenvector of $M$ corresponding to

$\lambda = \max_i L_{ii}$ of $M$.

Gershgorin's disk theorem $\Rightarrow$ spectral radius

$\rho(M) = \max_i L_{ii}$.

$\rho(M) = \max \{ |\lambda_1|, \ldots, |\lambda_n| \}$.

Digraph $G$ is strongly connected $\Rightarrow$ Digraph of $M$ is, too

[Digraph of $M$ is the digraph w/ node set $V=\{1, \ldots, n\}$

such that $\exists$ an edge from $j$ to $i$ iff $M_{ij} \neq 0$.]

$M$ is irreducible (not similar via permutation to a block upper triangular matrix)

Perron-Frobenius Theorem:

$M$ is irreducible $\Rightarrow V$ is positive $\Rightarrow$ all initial info states contribute to the the equilibrium $x_e$. 
If $v_i = \gamma^n + i$, then $x_e = \left( \sum_{i=1}^{n} x_i(0) \right) 1$. 

average consensus condition

- If digraph is strongly connected and balanced 
  ($\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$ for all $i$, or total weight of edges leaving $i$ = total weight of edges entering $i$) 
  \[ 1^T L = 0 \] (1 is a left eigenvector of $L$ assoc. with the simple 0 eigenvalue) 
  $\Rightarrow$ average consensus is achieved iff $G$ is strongly conn. and balanced.

- If $G$ is undirected, then ave. consensus is achieved iff $G$ is connected.
Convergence Analysis for Dynamic Communication Topologies

- Set of a robot's neighbors may change over time
  - Communication links may be unreliable
  - Neighbors visible to a robot may change as robots move toward/away from each other

- What are the conditions under which consensus algorithms converge under random switching of the network topology?

Consensus alg: \( \dot{x}(t) = -L(t)x(t) \) linear model

Solution: \( x(t) = \Phi(t,0)x(0) \)

- Can show that \( \Phi(t,0) \) is a row-stochastic matrix with positive diagonal entries for all \( t > 0 \).
  - A square nonnegative matrix \( M \) is row-stochastic if all of its row sums equal 1.
  - Also, \( M1 = 1 \) \( \Rightarrow \) eigenvalue is 1

- Consensus is achieved if \( \lim_{t \to \infty} \Phi(t,0) = \frac{1}{M}1u^T \) for some column vector \( u \).

Assume that network topology is piecewise constant over finite lengths of time (dwell times), which are bounded below by a positive constant.

Switching times: \( t_1, t_2, \ldots \)

Dwell times: \( T_j = t_{j+1} - t_j \)

- Consensus achieved if \( \lim_{j \to \infty} e^{-\frac{1}{2}T_j}e^{-\frac{1}{2}(t_{j-1})T_{j-1}} \ldots e^{-\frac{1}{2}(t_0)T_0} = \frac{1}{M}1u^T \)
Convergence analysis involves the study of as products of stochastic matrices, specifically ones that are indecomposable and aperiodic (SIA matrices), for which:
\[ \lim_{k \to \infty} M_k^k = \mathbb{I} v^T \text{ for some column vector } v. \]

Let \( M = \{M_1, M_2, \ldots, M_k\} \) be a finite set of SIA matrices for which every finite product \( M_{ij} M_{ij-1} \ldots M_{i1} \) is SIA.

\[ \Rightarrow \text{ For each } \infty \text{ sequence } M_{i1}, M_{i2}, \ldots \Rightarrow \text{ a column vector } \]
\[ v \text{ such that } \lim_{j \to \infty} M_{ij} M_{ij-1} \ldots M_{i1} = \mathbb{I} v^T. \]

# of potential network topologies is finite

\[ \Rightarrow \text{ Set of matrices } \{M_j \equiv e^{-L(t_j)(t_{j+1} - t_j)}\}_{j=1}^{\infty} \]
is finite if the \( T_j = t_{j+1} - t_j \) are drawn from a finite set.

These matrices are SIA \( \Rightarrow \) can show consensus for a particular set of robot nearest-neighbor rules and conditions on \( \Delta t \) if the union of undirected graphs is connected. (see * on next page.)

[union of graphs is a graph whose node and edge sets are the unions of the node and edge sets of all graphs]

More realistic assumption about \( T_j \): they are drawn from an \( \infty \) but bounded set.

Let \( M = \{M_1, M_2, \ldots\} \) be an infinite set of nxn SIA matrices and let \( N_\tau \) be the # of different types of these matrices (have 0 entries + positive entries in the same locations).

Define \( f(P) = 1 - \min_{i_1, i_2} \Sigma_j \min (P_{ij}, P_{i_2j}) \)
\[
\lim_{j \to \infty} M_{ij} M_{ij-1} \cdots M_{i1} = \pm \nu^T \quad \text{if} \quad \exists \, d \in [0,1)
\]

such that, for every \( W = M_{k1} M_{k2} \cdots M_{kn+1} \),

\[\lambda(W) \leq d.\]

[Satisfied if \( t \) an \( \infty \)-sequence of contiguous, uniformly bounded time intervals \( \Delta t \) such that across each interval, the union of the network graphs has a rooted directed spanning tree.]

[Satisfied if \( t \) an \( \infty \)-sequence of contiguous, uniformly bounded \( \Delta t \), having one of a finite number of different lengths, such that across each \( \Delta t \), the union of undirected network graphs is connected.]

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**Communication Delays and Asynchronous Consensus**

- Consider time delays \( \delta_{ij} \) for information communicated from robot \( j \) to reach robot \( i \).

Consensus algorithm becomes:

\[
\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t) [x_j(t - \delta_{ij}) - x_i(t - \delta_{ij})].
\]

If \( \delta_{ij} = \sigma \) and graph \( G \) is fixed, undirected, and connected, then average consensus is achieved iff

\[
0 \leq \sigma < \frac{\pi}{2\lambda_{\text{max}}(L)}
\]
Case where $\sigma_{ij}$ only affects info state being transmitted:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t) \left[ x_j(t-\sigma_{ij}) - x_i(t) \right]$$

If $\sigma_{ij} = 0$ and $G$ is directed and switching, the consensus result for a switching topology is valid for an arbitrary delay $\sigma$.

In an asynchronous consensus framework, robots exchange info at different times $t$ and update their states with possibly outdated info from neighbors.

Must consider heterogeneous robots, time-varying $\sigma_{ij}$, and communication packet dropout.

**Algebraic Connectivity and Spectral Properties of Graphs**

- Quantifies the convergence rate of consensus algorithms.

**Gershgorin theorem** ⇒ all $\lambda_i$'s of $L$ lie in a closed disk in the complex plane centered at $\Delta + 0j$ with radius $\Delta = \max_i d_i$, where $d_i$ is the degree of node $i$ (the # of edges incident to node $i$)

[Digraphs have indegrees and outdegrees.]

- Undirected graphs: $L$ is symmetric & has real $\lambda_i$'s ⇒

$$0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 2\Delta$$

If $G$ is connected, then $\lambda_2 > 0$. $\lambda_2$ = algebraic connectivity of $G$. 
- Directed graphs that are balanced and strongly connected.

Symmetric part of $\mathbb{L}$: $\mathbb{L}_S = \frac{1}{2} (\mathbb{L} + \mathbb{L}^T)$

A continuous-time consensus is globally exponentially reached with a speed $\geq \lambda_2(\mathbb{L}_S)$.

**Synchronization of Coupled Oscillators**

- Applications in physics, biology, neuroscience, math.
  - Synchronous flashing of fireflies
  - Chemical/biological oscillators
  - Networks of pacemaker cells in the heart

$\Theta_i =$ phase of $i$th oscillator, $i \in \{1, \ldots, N\}$

$\omega_i =$ natural frequency of $i$th oscillator

Nonlinear extension of consensus algorithm:

$$\dot{\Theta}_i = \frac{K}{N} \sum_{j \in N_i} \sin(\Theta_j - \Theta_i) + \omega_i$$

$K =$ coupling strength

- If $K$ is sufficiently large, then for a network with all-to-all edges, synchronization to the aligned state is globally achieved for all initial states.

- Can write this model as: (For the case where all $j$ are neighbors of each $i$)$$\dot{\Theta}_i = K r \sin(\Psi - \Theta_i) + \omega_i$$

where $r = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\Theta_j} \right|$

- $r$ measures phase coherence
- $\Psi$ is the average phase

[Diagram of swarm of particles on a unit circle in the complex plane]