Geometric Representations for Multi-Robot Systems

*Website for Distributed Control of Robotic Networks, by Bullo, Cortés, and Martínez (2009)*

- Deployment over a region
- Rendezvous at a common point
- Pattern formation
- Move in a synchronized manner
- Agents have no global knowledge of state of the network, can observe only their closest neighbors
- Focus on motion coordination algorithms with provably correct strategies
- Strong connection to certain geometric objects and geometric optimization problems
  - Proximity graphs, Voronoi cells, optimization problems induced by geometric objects

I. Basic Geometric Concepts

\[ P, q \in \mathbb{R}^d \]
- Closed segment:
\[ [P, q] = \{ \lambda P + (1-\lambda)q \mid \lambda \in [0,1] \} \]
- Closed halfspace of \( \mathbb{R}^d \) of points closer to \( P \) than \( q \):
  - (halfplane)
\[ H_{P,q} = \{ x \in \mathbb{R}^d \mid \| x - P \|_2 \leq \| x - q \|_2 \} \]

- Set in \( \mathbb{R}^d \):
\[ S \subset \mathbb{R}^d \]
- \( S \) is convex if for any \( P, q \in S \), \([P, q]\) is contained in \( S \).
- Convex hull of a set, \( \text{co}(S) \), is the smallest convex set that contains \( S \).
• \( S = \{ p_1, p_2, p_3 \} \)

• \( \text{co}(S) = \{ \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 \mid \lambda_i \geq 0, \sum \lambda_i = 1 \} \)

- Polygon: set in \( \mathbb{R}^2 \) whose boundary is the union of a finite number of closed segments.
  - Simple polytope: boundary is not self-intersecting

- Polytope: generalization of polygon to \( \mathbb{R}^d \), \( d \geq 3 \)
  - Convex polytope in \( \mathbb{R}^d \): convex hull of a finite set of points in \( \mathbb{R}^d \) / bounded intersection of a finite set of halfspaces

- A \( d-1 \) facet (or face) is the intersection between the polytope and the boundary of a closed halfspace that defines the polytope.

- Edges are facets of dimension 1
- Vertices are facets of dimension 0

- Partition of \( S \) is a collection of closed connected sets \( \{ W_1, W_2, ..., W_m \} \) for which:
  \( S = \bigcup_{i=1}^{m} W_i \) and \( \text{int}(W_j) \cap \text{int}(W_k) = \emptyset \) for \( j, k \in \{1, ..., m\} \).

- A Voronoi partition of a set \( S \) generated by a set of points \( P = \{ p_1, ..., p_n \} \) in \( S \) is defined as:
\[ V(P) = \{ V_i(P), \ldots, V_n(P) \} \], where
\[ V_i(P) = \{ \frac{\bar{z}}{f} \in S \mid \text{dist}(\bar{z}, f) \leq \text{dist}(\bar{z}, f) + \exists f_j \in P \setminus \{ f_i \} \}. \]

Voronoi cell of \( f_i \)
- Set of points of \( S \) that are closer to \( f_i \) than to any of the other points in \( P \).

Voronoi partition of the circle generated by 5 points:

* An \( r \)-limited Voronoi partition inside \( S \) is defined as: \( V_r(P) = \{ V_{i,r}(P), \ldots, V_{n,r}(P) \} \), where \( V_{i,r}(P) = V_i(P) \cap B(f_i, r) \)

closed ball in \( \mathbb{R}^2 \) centered at \( f_i \) with radius \( r \)

II. Proximity Graphs

* A proximity graph \( G \) at \( P = \{ f_1, \ldots, f_n \} \subset \mathbb{R}^d \), \( G(P) \), is an undirected graph with vertex set \( P \) and with edge set \( E_G(P) \subseteq \{ f_i, f_j \} \subset P \times P / f_i \neq f_j \}

- Edge set is a function of the relative locations of the points

* Different types of proximity graphs:

  1. \( r \)-disk graph: \( \{ f_i, f_j \} \in E_G(P) \) if \( \| f_i - f_j \| \leq r \).

  2. Delaunay graph: \( \ldots \) if \( V_i(P) \cap V_j(P) \neq \emptyset \).
3. r-limited Delaunay graph: \( \exists \{ i, j \} \in E_G(P) \) if 
\[ V_i \cap r(P) \cap V_j \cap r(P) \neq \emptyset \]

4. Visibility graph: "" if the closed segment in an environment \( \mathcal{Q} \subset \mathbb{R}^2 \)
\[ \{ i, j \} \subset \mathcal{Q} \]

5. Complete graph: all pairs of points are edges.
(fully connected graph)

Given a set \( P = \{ p_1, \ldots, p_n \} \subset \mathbb{R}^d \) and a proximity graph \( G \), the set of neighbors of \( f_i \) according to \( G \) is:
\[ N_{G, f_i}(P) = \{ q \in P \mid \{ f_i, q \} \in E_G(P) \} \]

III. Spatially Distributed Maps

- Given a set \( Y \) and a proximity graph \( G \), a map
\[ T: (\mathbb{R}^d)^n \rightarrow Y^n \] is spatially distributed over \( G \) if the \( j \)-th component of \( T_j \), \( T_j \), evaluated at any \( P = \{ p_1, \ldots, p_n \} \subset (\mathbb{R}^d)^n \) is a function only of \( f_j \) and of the vertices in \( G(P) \) that are neighbors of \( f_j \).
- Each agent \( j \) has sufficient information to compute \( T_j(P) \).

- Given proximity graphs \( G_1 \) and \( G_2 \), \( G_1 \) is spatially distributed over \( G_2 \) if each agent, when informed about the location of its neighbors according to \( G_2 \), has sufficient info to determine its set of neighbors according to \( G_1 \).
IV. Encoding Coordination Tasks

- Aggregate behavior of agents is evaluated using objective functions: achieving a coordination task = moving agents and changing their states to maximize or minimize an objective function.

- Formulate coordination objectives using functions from geometric optimization.

Deployment

- Place a network of mobile agents in a given environment to achieve maximum coverage (can be defined in different ways).

- Consider a convex polytope (environment) \( Q \subset \mathbb{R}^d \).

Density function \( \phi : Q \rightarrow [0, \infty) \)

- \( \phi \) quantifies the relative importance of different points in the environment (e.g., probability that an event of interest takes place)

Performance function \( f : [0, \infty) \rightarrow \mathbb{R} \) describes the utility of placing an agent at a certain distance from a given location in \( Q \) (agent)

- Smaller distance \( \rightarrow \) larger value of \( f \) decreases as \( \phi \)

- \( \phi \) increases as \( \rho \) moves away from \( q \) (harder to detect the sound)

Goal is to maximize the expected value of the coverage performance by agents in \( Q \subset \mathbb{R}^d \), given \( \phi \) and \( f \).
Define the objective function as $H : Q^n \rightarrow \mathbb{R}$

$$H(p) = \max_{i \in \{1, \ldots, n\}} \int_{Q} f(||q - p_i||) \phi(q) dq$$

$p = \{p_1, \ldots, p_n\}$ $H$ depends on all locations $p_i$. 

Want to find local maximizers of $H$ (set of $p_i$ that maximize its value).

- $f$ should be non-increasing, piecewise continuously differentiable function, possibly with finite jump discontinuities.

- Interpretation of $H$:
  - For each location $q \in Q$, consider the best coverage of $q$ among those provided by each of the agents $i, \ldots, n$. This is the value $\max_{i \in \{1, \ldots, n\}} f(||q - p_i||)$.
  - Evaluate the importance $\phi(q)$ of the location $q$.
  - Sum this quantity over all locations in $Q$ — this is $H(p)$, a measure of the overall coverage.

Can also define $H(p)$ in terms of the Voronoi partition of $Q$ generated by $p = \{p_1, \ldots, p_n\}$ (no repeated points $p_i$)

$$H(p) = \sum_{i=1}^{n} \int_{V_i(p)} f(||q - p_i||) \phi(q) dq$$

- It can be proved that the Voronoi partition $V(p)$ yields the maximum (optimal) value of $H(p)$ among all partitions of $Q$. 

- Could use $W_i$ from any partition.
The gradient of $H(P)$ is spatially distributed over the Delaunay graph.

**Application: Visibility-Based Deployment**

Nonconvex polytope $Q \subset \mathbb{R}^d$, $f \in \mathbb{Q}$

$S(f) = \{q \in Q | \exists f, f \in Q\}$ is the visible region in $Q$ from the location $f$.

$L_s(q) = \begin{cases} 1, & q \in S \\ 0, & \frac{q}{f} \notin S \end{cases}$ indicator function

$H_{vis}(P) = \int_{Q} \max_{i \in \{1, \ldots, n\}} L_s(p_i)(q) dq$

- In 2D, $H_{vis}$ measures the area of the subset of $Q$ composed of points that are visible from at least one of the agents located at $f_i$.
- A density $f: Q \rightarrow [0, \infty)$ could be included in $H_{vis}$ to assign varying levels of importance throughout the environment.

**Application: Rendezvous (a spatial version of consensus)**

Agreement over location of agents

$V_{diam}(P) = \max \{\|f_i - f_j\|_1 | i,j \in \{1, \ldots, n\}\}$ objective function

$= 0$ iff $f_i = f_j \forall i,j \in \{1, \ldots, n\}$. 
Application: Cohesion and Collision Avoidance

\[ H_{\text{coh}, G}(p) = \sum_{i \neq j} h(||r_i - r_j||) \quad G \text{ is a proximity graph} \]

\[ h : (0, \infty) \rightarrow \mathbb{R} \text{ is a repulsion/attraction function} \]

\[ h(r) \quad \text{(constant)} \quad \text{example } h \]
Designing Motion Coordination Algorithms

- Identical agents that can communicate
- Can design coordination algorithm from the objective function:
  1. Identify an objective fn. \( H(p) \) that is relevant to the desired coordination task.
  2. Analyze smoothness properties of \( H \) and compute its gradient or generalized gradient.
  3. Characterize the critical pts of \( H \), which encode the desired network configurations.
  4. Identify proximity graphs \( G(p) \) to facilitate computation of the gradient of \( H \) in a spatially distrib. manner: if at least one of these graphs is spatially dist. over the agents' communication graph, then a control law for each agent consists of following the gradient of \( H \).

- Closed-loop network trajectories will converge to set of critical pts of \( H \) (accordin to an invariance principle)

Execution of control laws:
- In each communication round, each agent: 1) transmits its position + receives its neighbor positions; 2) computes a notion of the geometric center of its own cell, determined according to some partition of the environment.
- Between commun. rounds, each robot moves toward this center.
Step 1 for different coordination tasks.

\[ H(P) = \sum_{i=1}^{n} \int_{V_i(P)} f(\|q - f_i\|_2) \phi(q) dq \quad P = \{f_1, \ldots, f_n\} \]

(a) Distortion problem: \[ f(x) = -x^2 \]

\[ H(P) = -\sum_{i=1}^{n} \int_{V_i(P)} \|q - f_i\|_2^2 \phi(q) dq \]

- In signal compression, \(-H\) is called the distortion function; "distortion" refers to the average deformation (weighted by \(\phi(q)\)) caused by reproducing \(q\) with the location \(f_i\), where \(q \in V_i(P)\).

(b) Area problem: \[ f(x) = \mathbb{1}_{[0,a]}(x) \], \(a > 0\)

\[ T_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases} \quad \text{indicator function} \]

\[ H(P) = \sum_{i=1}^{n} \int_{V_i(P)} T_S(\|q - f_i\|_2) \phi(q) dq \]

\[ = \sum_{i=1}^{n} \int_{V_i(P) \cap \overline{B}(f_i,a)} \phi(q) dq \]

\[ = \sum_{i=1}^{n} A_{\phi}(V_i(P) \cap \overline{B}(f_i,a)) \]

where \(A_{\phi}(S) = \int_S \phi(q) dq\) is the area of \(S\) weighted according to \(\phi(q)\).
$H(p) = A \bigcap_{i=1}^{n} B(p_i, a) = \text{area of the union of } n \text{ balls, weighted according to } \phi(q)$

**Step 2** for different coordination tasks

- Characterize the smoothness of $H(p)$

For a performance function $f : [0, \infty) \rightarrow \mathbb{R}$, let $\text{Dscn}(f)$ denote the (finite) set of points where $f$ is discontinuous.

For each $a \in \text{Dscn}(f)$, we define:

\[
\begin{align*}
    f_-(a) &= \lim_{x \to a^-} f(x), \\
    f_+(a) &= \lim_{x \to a^+} f(x)
\end{align*}
\]

\(x\) approaches from the left \( \xrightarrow{\text{from the left}} \) \( x \to a^- \)

\(x\) approaches from the right \( \xrightarrow{\text{from the right}} \) \( x \to a^+ \)

- Given a set $Q \subset \mathbb{R}^d$ that is bounded and measurable, a density $\phi : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$, and a performance function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, the objective fn. $H : Q^n \rightarrow \mathbb{R}$ is:

1. **Globally Lipschitz on $Q^n$**
   - Given $S \subset \mathbb{R}^k$, a function $f : S \rightarrow \mathbb{R}^k$ is globally Lipschitz if there exists $K > 0$ such that $\|f(x) - f(y)\|_2 \leq K \|x - y\|_2$ for all $x, y \in S$.

2. **Continuously differentiable on $Q^n \setminus \text{Pcoin}c$**

   $\text{Pcoin}c = \{ (f_1, \ldots, f_n) \in (\mathbb{R}^d)^n | f_i = f_j \text{ for some } i \neq j \}$

   where for $i \in \{1, \ldots, n\}$.
\[
\frac{\partial H}{\partial f_i}(p) = \int_{V_i(p)} \frac{\partial}{\partial f_i} f(\mathbf{q}) \cdot (-\mathbf{p} \cdot \mathbf{n}) \phi(q) dq \\
+ \sum_{a \in \text{Dscn}(f)} \left( f^-(a) - f^+(a) \right) \int_{V_i(p) \cap \partial B(f_i,a)} n_{\text{out}}(q) \phi(q) dq \\
\cdot n_{\text{out}} \text{ is the outward normal vector to } B(f_i,a)
\]

- The gradient of \( H \) is spatially distributed over the Delaunay graph (for which \( \{ f_i, p_j \} \in E_G(p) \text{ if } V_i(p) \cap V_j(p) \neq \emptyset \)).
- The motion of \( f_i \) affects \( V_i(p) \) and \( V_j(p) \), \( \{ f_i, p_j \} \in E_G(p) \).

- **Distortion problem**, \( f(x) = -x^2 \)
- \( f(x) \) has no discontinuities \( \Rightarrow \) 2nd term in \( \frac{\partial H}{\partial f_i} \) is zero
- Centroid of \( V_i(p) \) with respect to \( \phi \):
  \[
  CM_\phi(V_i(p)) = \frac{1}{A_\phi(V_i(p))} \int_{V_i(p)} \phi(q) dq \\
  A_\phi(V_i(p)) = \int_{V_i(p)} \phi(q) dq
  \]
- Area of \( V_i(p) \) weighted according to \( \phi \):
  \[
  A_\phi(V_i(p)) = \int_{V_i(p)} \phi(q) dq
  \]
- Polar moment of inertia of \( V_i(p) \) about \( f_i \in V_i(p) \):
  \[
  \int_\phi(V_i(p), f_i) = \int_{V_i(p)} \| \mathbf{q} - \mathbf{p} \|^2 \phi(q) dq
  \]
Parallel axis theorem:
\[ J_\phi(V_i(P), \phi_i) = J_\phi(V_i(P), CM_\phi(V_i(P))) + A_\phi(V_i(P)) \| \phi_i - CM_\phi(V_i(P)) \| ^2 \]

Note that for this problem,
\[ H(P) = - \sum_{i=1}^{n} J_\phi(V_i(P), \phi_i) \]

- It can be shown that:
\[ \frac{\partial H}{\partial \phi_i}(P) = 2 A_\phi(V_i(P)) (CM_\phi(V_i(P)) - \phi_i) \]

\[ \Rightarrow \text{the } i\text{th component of the gradient points in the direction of the vector from } \phi_i \text{ to the centroid of its Voronoi cell.} \]

(b) Area problem, \( f(x) = 1_{[0,a]}(x), \ a > 0 \)

- \( f(x) \) is differentiable everywhere except at the discontinuity \( x = a \), and its derivative is 0.

\[ \Rightarrow \text{1st term in } \frac{\partial H}{\partial \phi_i} \text{ is zero} \]

\[ \Rightarrow \frac{\partial H}{\partial \phi_i}(P) = (f(a) - f_+(a)) \int_{V_i(P) \cap \partial B(\phi_i,a)} \omega_0(q) \phi(q) dq \]

This gradient is the average of the normal at each point of \( V_i(P) \cap \partial B(\phi_i,a) \).

Ex) \( \phi(q) \) is constant:

By moving along the gradient directions (arrows), the agents decrease overlap and cover new regions of space.
Step 3 for different coordination tasks

(a) Distortion problem:
- Critical points of \( f(p) \) are the set of centroidal Voronoi configurations in \( Q \): each point is the centroid of its own Voronoi cell \( f_i = \text{CM} \phi(V_i(p)) \).

(b) Area problem:
- Critical points of \( H(p) \) are the set of \( a \)-limited area-centered Voronoi configurations in \( Q \): each \( f_i \) is a local maximum for the area of \( V_i(p) \cap \overline{B(f_i,a)} \) at fixed \( V_i(p) \).

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Step 4

(a) Distortion problem: The gradient of \( H \) is spatially distributed over the Delaunay graph.

(b) Area problem: The gradient of \( H \) is spatially distributed over the \( 2a \)-limited Delaunay graph \( \{ f_i, f_j \} \in \mathcal{E} \cap \mathcal{G}(p) \) if \( V_i, a(p) \cap V_j, a(p) \neq \emptyset \).

- Robots with range-limited interactions can compute the gradients of \( H \).

Coordination algorithm:

(a) Move toward the centroid of own Voronoi cell [Distortion]

(b) Move in the direction of the weighted normal to the boundary of own cell (Area)
Algorithms for Coverage Control

- Definition of a robotic network $S = (I, R, E_{cm})$:
  1. $I = \{1, ..., n\}$: set of unique identifiers
  2. $R = \{R^i \mid i \in I\} = \{(X^i, U^i, X_0^i, f^i)\}_{i \in I}$ is a set of mobile robots, where:
     - $X^i = \text{d-dimensional state space}$
     - $U^i \subseteq \mathbb{R}^m$, $0 \in U^i$; $U^i$ = input space
     - $X_0^i$ = set of allowable initial states; $X_0^i \subseteq X^i$
     - $f^i : X^i \times U^i \rightarrow \mathbb{R}^d$ is a continuously differentiable control vector field on $X^i$ that determines the robot motion according to:
       - $\dot{x}(t) = f(x(t), u(t))$
       - $x(t)$ = physical state of robot (ex. $\begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix}$)
       - $u(t)$ = control input
  3. $E_{cm}$ = communication edge map: map from $X^1 \times X^2 \times ... \times X^n$ to the subsets of $I \times I$.

- If all robots are identical, then the robot network is uniform.

- $G^c = (I, E_{cm}) = (V, E)$, the communication graph of the network. This is defined by a proximity graph (defined earlier).
Planar models for robots, $\dot{x}(t) = f(x(t), u(t))$

$\mathbf{x} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$

$\dot{x} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v \cos \theta \\ v \sin \theta \\ \omega \end{bmatrix}$

$v = \text{forward linear velocity}$
$\omega = \text{angular velocity}$

$u = \begin{bmatrix} v \\ \omega \end{bmatrix}$

Different types of models:

1. Unicycle: $v, \omega \in [-1, 1]$
2. Differential drive robot:
   
   $v = \frac{1}{2} (v_{\text{right}} + v_{\text{left}}), \quad \omega = \frac{1}{L} (v_{\text{right}} - v_{\text{left}})$

   $v_{\text{right}}, v_{\text{left}} \in [-1, 1]$
3. Reeds-Shepp car: $v \in \{-1, 0, 1\}, \quad \omega \in [-1, 1]$
4. Dubins vehicle: $v = 1, \omega \in [-1, 1]$

For the 4-wheeled robot:

- $(x, y)$ is midpoint of rear axle
- $\theta$ is orientation of rear axle
  - $v = \text{forward linear velocity of rear axle}$

- $\omega = \frac{v}{L} \tan \phi$, $\phi$ is vehicle steering angle
Examples of robotic networks

1. \( S_D \): uniform network of robots moving according to the 1st-order model: \( \dot{x} = u \), \( u \in [-\text{u max}, \text{u max}]^d \), \( x \in \mathbb{R}^d \) (omnidirectional robot)
   - can move in any direction

   - Each robot can sense its own position and communicate with its neighbors, as defined by the Delaunay graph.

2. \( S_{LD} \): same as \( S_D \), except the communication graph is the \( r \)-limited Delaunay graph.

3. \( S_{vehicles} \): uniform network of robots moving according to:
   \[
   \begin{bmatrix}
   \dot{x} \\
   \dot{y} \\
   \dot{\theta}
   \end{bmatrix} =
   \begin{bmatrix}
   v \cos \theta \\
   v \sin \theta \\
   \omega
   \end{bmatrix}
   \]
   - Each robot can sense its own position.
   - Communication graph is the Delaunay graph.

Execution of control laws:

In each communication round, each agent:

1. Transmits its position to receive its neighbors' positions.
2. Computes a notion of the geometric center of its own cell, determined according to some partition of the environment \( Q \) (\( S_D, S_{LD}: Q \) is a polytope; \( S_{vehicles}: Q \) is a convex polygon)

Between comm. rounds, each robot moves toward this center.
Control law for distortion problem on network $S_D$

\[
\begin{align*}
V &= Q \cap (\cap \{ H_p \mid \text{prevd} \}) \\
\mathbf{\dot{p}} &= C M \phi(V) - \mathbf{p} \\
\text{Use: } \mathbf{p}(t+1) &= \mathbf{p}(t) + \mathbf{u}(t) \quad \text{(discrete-time motion model)}
\end{align*}
\]

$H_p$ is the half-space of points $q \in \mathbb{R}^d$ with the property that $\| q - p \|_2 \leq \| q - x \|_2$.

Control law for distortion problem on $S$-vehicles

\[
\begin{align*}
V &= Q \cap (\cap \{ H_p \mid \text{prevd} \}) \\
\mathbf{\dot{v}} &= K \begin{bmatrix} \cos \Theta & \sin \Theta \end{bmatrix} \cdot (\mathbf{p} - C M \phi(V)) \\
\omega &= 2K \tan^{-1} \left( \frac{[\sin \Theta \cos \Theta] \cdot (\mathbf{p} - C M \phi(V))}{[\cos \Theta \sin \Theta] \cdot (\mathbf{p} - C M \phi(V))} \right)
\end{align*}
\]

$k \in (0, \frac{1}{\max[\pi, \text{diam}(Q)]})$ so that $V, \omega \in [-1, 1]$ (can be implemented in unicycle and differential drive models)

\[\text{diam}(Q) = \text{maximum distance between any 2 points in } Q\]