Hybrid Systems

• A set of discrete modes, each with different continuous dynamics; extends finite state machines to include dynamics.
  - Can use reachability analysis to approximate the system behavior over time and address the reachability problem:
    “Is there a trajectory from some initial state to some target state? (\(x_0\) \(\rightarrow\) \(x_f\)).
  
  This problem can be solved algorithmically in a finite # of steps (that is, the reachability problem is decidable) when the continuous dynamics:
  
  a) are constant (timed + multirate automata)
  b) take values in a constant interval (rectangular automata)
  c) are a certain class of linear systems.

  (“A New Class of Decidable Hybrid Systems,” HSCC ’99)

• If the reachability problem is not decidable for a system, then you can compute an overapproximation of the reachable set:

  (1) Indirect method: develop a discrete abstraction of the hybrid system (can partition the continuous state space into finite # of sets)
  (2) Direct method: directly calculate the reachable set on the state space
Notation for a hybrid system

"An introduction to hybrid dynamical systems", Van der Schaft and Schumacher, 2000

\[ H = (L, X, A, W, E, Inv, Act) \]

- \( L \): set of discrete locations ("modes") — the vertices of a graph
- \( X \): continuous state space in which the continuous state variables take values
- \( A \): set of symbols that label the transitions (or "events")
- \( W \): continuous space of external variables \( w \)
- \( E \): finite set of transitions between locations — the edges of a graph
  - Each transition is defined by \((l, a, Guardl', Jumpa, l')\)
    \[ l, l' \in L, \ a \in A, \ Guardl' \subseteq X \]
    The transition from \( l \) to \( l' \) is enabled when the state \( x \) is in \( Guardl' \), and \( x \) is reset to a new value given by \( Jumpa \subseteq X \times X \).
- \( Inv \): maps the locations to subsets of \( X \) such that when the system is at location \( l \), \( x \in Inv(l) \), the "location invariant" of \( l \).
- \( Act \): a map that assigns a continuous vector field to each location, \( \dot{x} = f(l)(x, w) \).
  The solutions of the eq. are called the "activities" of the location \( l \).
\[
\begin{align*}
\dot{x} &= f(l)(x,w) \\
\text{Guard: } x(t) \geq \Delta &
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= f(l_2)(x,w) \\
\text{Jump: } x(t) \in \text{Inv}(l_2) &
\end{align*}
\]

- Trajectory of a hybrid system:
  An infinite sequence of continuous trajectories
  \((l_0, \delta_0, x_0, w_0) \xrightarrow{a_0} (l_1, \delta_1, x_1, w_1) \xrightarrow{a_1} (l_2, \delta_2, x_2, w_2) \xrightarrow{a_2} \ldots\)
  such that at the event times
  \(t_0 = \delta_0, \ t_1 = \delta_0 + \delta_1, \ t_2 = \delta_0 + \delta_1 + \delta_2, \ldots\)
  we have that:
  \[
  \begin{cases}
  x_j(t_j) \in \text{Guard } l_j, l_{j+1} & \forall j = 0, 1, 2, \ldots \\
  (x_j(t_j), x_{j+1}(t_j)) \in \text{Jump } l_j, l_{j+1}.
  \end{cases}
  \]

Here, \((l, \delta, x, w)\) is a continuous trajectory associated with location \(l\):
- \(\delta\) = nonnegative time
- \(w: [0, \delta] \to W\) is a piecewise constant function
- \(x: [0, \delta] \to X\) is a continuous, piecewise differentiable function such that:
  \[
  \begin{align*}
  x(t) &\in \text{Inv}(l) \quad \forall t \in (0, \delta), \\
  \dot{x}(t) &= f(l)(x,w) \quad \forall t \in (0, \delta) \text{ except for points of discontinuity of } w.
  \end{align*}
  \]