Bilinear Controllability of a Class of Advection-Diffusion-Reaction Systems

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***Abstract*—In this paper, we investigate the exact controllability properties of an advection-diffusion equation on a bounded domain, using time- and space-dependent velocity fields as the control parameters. This partial differential equation (PDE) is the Kolmogorov forward equation for a reflected diffusion process that models the spatiotemporal evolution of a swarm of agents. We prove that if a target probability density has bounded first-order weak derivatives and is uniformly bounded from below by a positive constant, then it can be reached in finite time using control inputs that are bounded in space and time. We then extend this controllability result to a class of advection-diffusion-reaction PDEs that corresponds to a hybrid-switching diffusion process (HSDP), in which case the reaction parameters are additionally incorporated as the control inputs. For the HSDP, we first constructively prove controllability of the associated continuous-time Markov chain (CTMC) system, in which the state space is finite. Then we show that our controllability results for the advection-diffusion equation and the CTMC can be combined to establish controllability of the forward equation of the HSDP. Lastly, we provide constructive solutions to the problem of asymptotically stabilizing an HSDP to a target non-negative stationary distribution using time-independent state feedback laws, which correspond to spatially-dependent coefficients of the associated system of PDEs.**

***Index Terms*—Swarm robotics, advection-diffusion-reaction PDE, stochastic processes, controllability, continuous-time Markov chains.**

I. Introduction

In recent years, there has been an extensive amount of work on control of large-scale multi-agent systems. A particular instance of these control problems is when the agents represent autonomous robots that must collectively achieve some desired global objective. This has motivated various investigations on the appropriate technical framework for modeling and control of such systems, called *robotic swarms*. One such framework involves modeling swarms using probability densities and controlling them through mean-field models [10], [28], [32]. Applications of these models to the control of robotic swarms has led to new questions from a control-theoretical point of view. One fundamental question is that of controllability: *given a mean-field model with a control parameter, what is the class of reachable probability densities?* Stated as such, the problem is not new since the use of mean-field models in the natural sciences and engineering is classical. However, the choice of control parameters in the context of swarm control has led to some non-classical controllability problems.

As one of the main contributions of this paper, we consider such a controllability problem for a robotic swarm that is described by a mean-field model in the form of an *advection-diffusion* partial differential equation (PDE). Similar controllability problems have been addressed previously in the literature. For example, motivated by problems arising from quantum physics, Blaquiere [8] used techniques from stochastic control to study a controllability problem in which a stochastic process evolves on a n-dimensional Euclidean space $\mathbb{R}^n$. A similar controllability result was proved in [13]. In [37], Porretta addressed a controllability problem for a Fokker-Planck equation evolving on the n-dimensional torus, along with an associated *mean-field game* problem [6]. This work applied observability inequalities that are typically used in PDE controllability problems. The results in [8], [13] were extended to a more general setting in which the stochastic process is a linear control system perturbed by a diffusion process [11]. Controllability problems for systems with a similar structure have also been considered in work on *multiplicative control* of PDEs [24].

In contrast to these works, in this paper, the stochastic process that models the agents' motion is confined to a bounded subset of a Euclidean space. Boundedness of the domain is a common constraint in many problems in swarm robotics, e.g. in [16], [32], where optimal control techniques were used to optimize swarm behavior. Additionally, the results in previous controllability studies were proven with control parameters that are square-integrable. However, in bilinear optimal control of PDEs associated with stochastic processes, the boundedness of the vector fields is a common requirement [16], [19], [20]. Toward this end, we establish controllability with control inputs that are (essentially) bounded in space and time.

Another contribution of this paper is our analysis of a controllability problem for the forward equation of a class of hybrid switching diffusion processes (HSDPs) [40]. These processes can be used as models for robots that switch between multiple behavioral states, e.g. [32], [16]. Our result is based on a controllability result for the forward equation of a related class of continuous-time Markov chains (CTMCs). A nontrivial issue in the problem of controlling the forward equation of CTMCs is the fact that the control parameters, which correspond to the transition rates of the Markov chain, are constrained to be positive. Hence, classical results on controllability of nonlinear control systems governed by ordinary differential equations (ODEs) do not apply. In spite of

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this issue, we prove controllability of these forward equations using piecewise constant control inputs. This controllability property can be attributed to the strong connectivity of the associated graphs.

As a final contribution of this paper, we consider the problem of stabilizing HSDPs to desired stationary distributions using time-independent and spatially-dependent controls or state feedback laws. A similar problem was considered in [30] for general controllable systems on unbounded domains with a single discrete state.

The results presented in this paper are partially extensions of our prior work in [15], where controllability was proved for the case in which a swarm modeled using a Fokker-Planck equation evolves on a one-dimensional bounded domain. Here, we generalize this result to multi-dimensional domains with sufficiently smooth boundaries that satisfy the *chain condition* (see Definition II.3). Additionally, the requirement in [15] for target densities to have second-order partial derivatives that are essentially bounded in space is relaxed to the requirement that only first-order partial derivatives must be bounded. The controllability result for the forward equation of CTMCs was first summarized in our prior work [14] without proof. A complete proof of this result is given in this paper. The controllability and stabilization results presented in this paper for the case of HSDPs are new.

The paper is organized as follows. In Section II, we establish notation and provide some definitions that are used throughout the paper. In Section III, we formulate the main problems that are addressed in the paper. In Section IV, we present a detailed analysis of the controllability properties of the systems defined in Section III. In Section V, we consider the problem of stabilizing target equilibrium densities of the system of PDEs using time-independent and spatially-dependent coefficients of the PDEs. Finally, we discuss conclusions in Section VI.

II. NOTATION

We denote the n-dimensional Euclidean space by $\mathbb{R}^n$. $\mathbb{R}^{n\times m}$ refers to the space of $n \times m$ matrices, and $\mathbb{R}_+$ refers to the set of non-negative real numbers. Given a vector $\mathbf{x} \in \mathbb{R}^n$, x_i denotes the i^{th} coordinate value of $\mathbf{x}$. For a matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$, A^{ij} refers to the element in the i^{th} row and j^{th} column of $\mathbf{A}$. For a subset $B \subset \mathbb{R}^M$, $\mathrm{int}(B)$ refers to the interior of the set B. $\mathbb{C}$, $\mathbb{C}_-$, and $\bar{\mathbb{C}}_-$ denote the set of complex numbers, the set of complex numbers with negative real parts, and the set of complex numbers with non-positive real parts, respectively. $\mathbb{Z}_+$ refers to the set of positive integers. We denote by Ω an open, bounded, and connected subset of a Euclidean domain $\mathbb{R}^n$. The boundary of Ω is denoted by $\partial\Omega$.

Definition II.1. *We will say that Ω is a $C^{1,1}$ **domain** if each point $\mathbf{x} \in \partial\Omega$ has a neighborhood $\mathcal{N}$ such that $\Omega \cap \mathcal{N}$ is represented by the inequality $x_n < \gamma(x_1, ..., x_{n-1})$ in some Cartesian coordinate system for some function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ that is at least once differentiable and has derivatives of order 1 that are Lipschitz continuous.*

For each $1 \le p < \infty$, we define $L^p(\Omega)$ as the Banach space of complex-valued measurable functions over the set Ω whose absolute value raised to p^{th} power has finite integral. We define $L^\infty(\Omega)$ as the space of essentially bounded measurable functions on Ω. The space $L^\infty(\Omega)$ is equipped with the norm $\|z\|_\infty = \text{ ess sup}_{\mathbf{x}\in\Omega}|z(\mathbf{x})|$, where ess $\sup_{\mathbf{x}\in\Omega}(\cdot)$ denotes the *essential supremum* attained by its argument over the domain Ω. The space $L^2(\Omega)$ is a Hilbert space when equipped with the standard inner product, $\langle\cdot,\cdot\rangle_2 : L^2(\Omega)\times L^2(\Omega) \to \mathbb{C}$, given by $\langle f, g\rangle_2 = \int_\Omega f(\mathbf{x})\bar{g}(\mathbf{x})d\mathbf{x}$ for each $f, g \in L^2(\Omega)$, where $\bar{g}$ is the complex conjugate of the function g. The norm $\|\cdot\|_2$ on the space $L^2(\Omega)$ is defined as $\|f\|_2 = \langle f, f\rangle_2^{1/2}$ for each $f \in L^2(\Omega)$. For a function $f \in L^2(\Omega)$ and a given constant c, we write $f \ge c$ to imply that f is real-valued and $f(\mathbf{x}) \ge c$ for almost every (a.e.) $\mathbf{x} \in \Omega$.

Let f_{x_i} denote the first-order (weak) partial derivative of the function f with respect to the coordinate x_i. Similarly, $f_{x_ix_i}$ will denote the second-order partial derivative of the function f with respect to the coordinate x_i. We define the Sobolev space $H^1(\Omega) = \left\{f \in L^2(\Omega) : f_{x_i} \in L^2(\Omega) \text{ for } 1 \le i \le N\right\}$. We equip this space with the usual Sobolev norm $\|\cdot\|_{H^1}$, given by $\|f\|_{H^1} = \left(\|f\|_2^2 + \sum_{i=1}^n \|f_{x_i}\|_2^2\right)^{1/2}$ for each $f \in H^1(\Omega)$. The weak gradient of a function $f \in H^1(\Omega)$ will be denoted by $\nabla f = [f_{x_1} \ ... \ f_{x_n}]^T$.

Definition II.2. *We will call Ω an **extension domain** if there exists a linear bounded operator $E : H^1(\Omega) \to H^1(\mathbb{R}^n)$ such that $(Ef)(\mathbf{x}) = f(\mathbf{x})$ for a.e. $\mathbf{x} \in \Omega$.*

An example of an extension domain is a domain with Lipschitz boundary [1][Theorem 10.4.1]. Unless otherwise stated, the **default assumption** in this paper will be that Ω **is an extension domain**. The exponential stability results will only require this default assumption. However, to prove the controllability result, we will need the stronger assumption that the domain Ω is $C^{1,1}$ or convex. An additional assumption about the domain Ω will be needed to prove the controllability result, which motivates the following definition.

Definition II.3. *The domain Ω will be said to satisfy the **chain condition** if there exists a constant $C > 0$ such that for every $\mathbf{x}, \bar{\mathbf{x}} \in \Omega$ and every positive $n \in \mathbb{Z}_+$, there exists a sequence of points $\mathbf{x}_i \in \Omega$, $0 \le i \le n$, such that $\mathbf{x}_0 = \mathbf{x}$, $\mathbf{x}_n = \bar{\mathbf{x}}$, and $|\mathbf{x}_i - \mathbf{x}_{i+1}| \le \frac{C}{n}|\mathbf{x} - \bar{\mathbf{x}}|$ for all $i = 0, ..., n-1$. Here $|\cdot|$ denotes the standard Euclidean norm.*

Note that every convex domain satisfies the chain condition. For a given real-valued function $a \in L^\infty(\Omega)$, $L^2_a(\Omega)$ refers to the set of all functions f such that $\int_0^1 |f(\mathbf{x})|^2 a(\mathbf{x})d\mathbf{x} < \infty$. We will always assume that the associated function a is uniformly bounded from below by a positive constant, in which case the space $L^2_a(\Omega)$ is a Hilbert space with respect to the weighted inner product $\langle\cdot,\cdot\rangle_a : L^2_a(\Omega)\times L^2_a(\Omega) \to \mathbb{R}$, given by $\langle f, g\rangle_a = \int_\Omega f(\mathbf{x})\bar{g}(\mathbf{x})a(\mathbf{x})d\mathbf{x}$ for each $f, g \in L^2_a(\Omega)$. We will also need the space $H^1_a(\Omega) = \left\{z \in L^2_a(\Omega) : (az)_{x_i} \in L^2(\Omega) \text{ for } 1 \le i \le N\right\}$, equipped with the norm $\|f\|_{H^1_a} = \left(\|f\|_a^2 + \sum_{i=1}^n \|(af)_{x_i}\|_2^2\right)^{1/2}$. When $a = \mathbf{1}$, where $\mathbf{1}$ is the function that takes the value 1 a.e. on Ω, the spaces $L^1(\Omega)$ and $H^1(\Omega)$ coincide with the spaces $L^1_a(\Omega)$ and $H^1_a(\Omega)$, respectively. We will also need the spaces $W^{1,\infty}(\Omega) = \left\{z \in\right.$

$L^\infty(\Omega) : z_{x_i} \in L^\infty(\Omega) \text{ for } 1 \leq i \leq N\}$ and $W^{2,\infty}(\Omega) = \{z \in W^{1,\infty}(\Omega) : z_{x_i x_i} \in L^\infty(\Omega) \text{ for } 1 \leq i \leq N\}$. Let X be a Hilbert space with the norm $\|\cdot\|_X$. The space $C([0,T];X)$ consists of all continuous functions $u : [0,T] \to X$ for which $\|u\|_{C([0,T];X)} := \max_{0\leq t\leq T} \|u(t)\|_X < \infty$. If Y is a Hilbert space, then $\mathcal{L}(X,Y)$ will denote the space of linear bounded operators from X to Y. We will also use the multiplication operator $\mathcal{M}_a : L^2(\Omega) \to L^2(\Omega)$, defined as $(\mathcal{M}_a u)(\mathbf{x}) = a(\mathbf{x})u(\mathbf{x})$ for a.e. $\mathbf{x} \in \Omega$ and each $u \in L^2(\Omega)$.

We will need an appropriate notion of a solution of the PDE (5). Toward this end, let A be a closed linear operator that is densely defined on a subset $\mathcal{D}(A)$, the domain of the operator, of a Hilbert space X. We will define $\text{spec}(A)$ as the set $\{\lambda \in \mathbb{C} : \lambda\mathbb{I} - A \text{ is not invertible in } X\}$, where $\mathbb{I}$ is the identity map on X. If A is a bounded operator, then $\|A\|_{op}$ will denote the operator norm induced by the norm defined on X. From [17], we have the following definition.

Definition II.4. *For a given time $T > 0$, a **mild solution** of the ODE*

$$\dot{u}(t) = Au(t); \;\; u(0) = u_0 \in X \tag{1}$$

is a function $u \in C([0,T];X)$ such that $u(t) = u_0 + A\int_0^t u(s)ds$ for each $t \in [0,T]$.

Under appropriate conditions satisfied by A, the mild solution is given by a *strongly continuous semigroup* of linear operators, $(\mathcal{T}(t))_{t\geq 0}$, that are *generated* by the operator A [17].

The differential equations that we analyze in this paper will be non-autonomous in general. Hence, we must adapt the notion of a mild solution to these types of equations.

Definition II.5. *Let A_i be a closed linear operator with domain $\mathcal{D}(A_i)$ for each $i \in \mathbb{Z}_+$. Suppose that for a certain time interval $[0,T]$, a piecewise constant family of operators is given by a map, $t \mapsto A(t)$, for which there exists a partition $[0,T] = \cup_{i\in\mathbb{Z}_+}[a_i, a_{i+1})$ such that $a_i \leq a_{i+1}$ for each $i \in \mathbb{Z}_+$ and $A(t) = A_i$ for each $t \in [a_i, a_{i+1})$. Then a mild solution of the ODE*

$$\dot{u}(t) = A(t)u(t); \;\; u(0) = u_0 \in X \tag{2}$$

is a function $u \in C([0,T];X)$ such that

$$u(t) = u_0 + \sum_{i\in\mathbb{Z}_+} A_i \int_{\min\{t,a_i\}}^{\min\{t,a_{i+1}\}} u(s)ds \tag{3}$$

for each $t \in [0,T]$.

There is in fact a more general notion of mild solutions that arises from two-parameter semigroups of operators generated by time-varying linear operators. However, the definition (3) will be sufficient for our purposes, since one can construct solutions of the ODE (2) by treating it as an autonomous system in each time interval $[a_i, a_{i+1})$ and concatenating these solutions together to obtain the solution u. Note that the mild solution is defined with respect to an operator A or collection of operators $A(t)$; when we refer to such a solution, the associated operator(s) will be clear from the context. We will also need the notion of a positive semigroup, which is defined as follows.

Definition II.6. *A strongly continuous semigroup of linear operators $(\mathcal{T}(t))_{t\geq 0}$ on a Hilbert space X is called **positive** if $u \in X$ such that $u \geq 0$ implies that $\mathcal{T}(t)u \geq 0$ for all $t \geq 0$.*

We introduce some additional notation from graph theory which will be used in the coming sections. We denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a directed graph with a set of M vertices, $\mathcal{V} = \{1, 2, ..., M\}$, and a set of $N_\mathcal{E}$ edges, $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$. An edge from vertex $i \in \mathcal{V}$ to vertex $j \in \mathcal{V}$ is denoted by $e = (i,j) \in \mathcal{E}$. We define a source map $S : \mathcal{E} \to \mathcal{V}$ and a target map $T : \mathcal{E} \to \mathcal{V}$ for which $S(e) = i$ and $T(e) = j$ whenever $e = (i,j) \in \mathcal{E}$. There is a *directed path* of length s from vertex $i \in \mathcal{V}$ to vertex $j \in \mathcal{V}$ if there exists a sequence of edges $\{e_i\}_{i=1}^s$ in $\mathcal{E}$ such that $S(e_1) = i$, $T(e_s) = j$, and $S(e_k) = T(e_{k-1})$ for all $2 \leq k \leq s$. A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called *strongly connected* if for every pair of distinct vertices v_0, $v_T \in \mathcal{V}$, there exists a *directed path* of edges in $\mathcal{E}$ connecting v_0 to v_T. We assume that $(i,i) \notin \mathcal{E}$ for all $i \in \mathcal{V}$. The graph $\mathcal{G}$ is said to be *bidirected* if $e \in \mathcal{E}$ implies that $\tilde{e} = (T(e), S(e))$ also lies in $\mathcal{E}$.

III. PROBLEM FORMULATION

In this section, we will formulate the main problems that are addressed in this paper. The precise regularity conditions under which we study the solutions and parameters of the PDEs are presented in Section IV. On the other hand, the associated systems of stochastic differential equations (SDEs) will be treated in this paper only in a formal manner, in order to motivate the PDE controllability problem that we investigate. We note that one can prove existence of the stochastic processes that we consider by extending the semigroup-theoretic arguments in this paper to the theory of *Dirichlet forms* [5].

Consider a swarm of N_p agents that are deployed on the n-dimensional domain Ω. The position of each agent, indexed by $i \in \{1, 2, ..., N_p\}$, evolves according to a stochastic process $\mathbf{Z}_i(t) \in \Omega$, where t denotes time. We assume that the agents are non-interacting. Therefore, the random variables that correspond to the dynamics of each agent are independent and identically distributed, and we can drop the subscript i and define the problem in terms of a single stochastic process $\mathbf{Z}(t) \in \Omega$. The deterministic motion of each agent is defined by a velocity vector field $\mathbf{v}(\mathbf{x}, t) \in \mathbb{R}^n$, where $\mathbf{x} \in \Omega$. This motion is perturbed by a n-dimensional *Wiener process* $\mathbf{W}(t)$, which models noise. This process can be a model for stochasticity arising from inherent sensor and actuator noise. Alternatively, noise could be actively programmed into the agents' motion to implement more exploratory agent behaviors and to take advantage of the smoothening effect of the process on the agents' probability densities. Given the parameter $\mathbf{v}(\mathbf{x}, t)$, each agent evolves according to a *reflected diffusion process* $\mathbf{Z}(t)$ that satisfies the following SDE [36]:

$$\begin{aligned} d\mathbf{Z}(t) &= \mathbf{v}(\mathbf{Z}(t), t)dt + \sqrt{2D}d\mathbf{W}(t) + \mathbf{n}(\mathbf{Z}(t))d\psi(t), \\ \mathbf{Z}(0) &= \mathbf{Z}_0, \end{aligned} \tag{4}$$

where $\psi(t) \in \mathbb{R}$ is called the *reflecting function* or *local time* [5], [36], a stochastic process that constrains $\mathbf{Z}(t)$ to the domain Ω; $\mathbf{n}(\mathbf{x})$ is the normal to the boundary at $\mathbf{x} \in \partial\Omega$; and

$D > 0$ is the diffusion constant. Without loss of generality, we assume that $D = 1$.

We now pose the problem of determining the existence of the robot control law, defined as the velocity field $\mathbf{v}(\cdot, t)$, that drives the swarm to a target spatial distribution over the domain.

Problem III.1. *Given a time $t_f > 0$ and a target probability density $f : \Omega \to \mathbb{R}_+$ such that $\int_\Omega f(\mathbf{x})d\mathbf{x} = 1$, determine if there exists a feedback control law $\mathbf{v} : \Omega \times [0, t_f] \to \mathbb{R}^n$ such that the process* (4) *satisfies $\mathbb{P}(\mathbf{Z}(T) \in \Gamma) = \int_\Gamma f(\mathbf{x})d\mathbf{x}$ for each measurable subset $\Gamma \subset \Omega$.*

The *Kolmogorov forward equation* corresponding to the SDE (4) is given by:

$$\begin{aligned} y_t = \Delta y - \nabla \cdot (\mathbf{v}(\mathbf{x}, t)y) \quad & in \quad \Omega \times [0, T] \\ y(\cdot, 0) = y^0 \quad & in \quad \Omega \\ \mathbf{n} \cdot (\nabla y - \mathbf{v}(\mathbf{x}, t)y) = 0 \quad & in \quad \partial\Omega \times [0, T]. \end{aligned} \tag{5}$$

The solution $y(\mathbf{x}, t)$ of this equation represents the probability density of a single agent occupying position $\mathbf{x} \in \Omega$ at time t, or alternatively, the density of a population of agents at this position and time. The PDE (5) is related to the SDE (4) through the relation $\mathbb{P}(\mathbf{Z}(t) \in \Gamma) = \int_\Gamma y(\mathbf{x}, t)d\mathbf{x}$ for all $t \in [0, t_f]$ and all measurable $\Gamma \subset \Omega$. Therefore, the solution $y(\mathbf{x}, t)$ captures the *mean-field* behavior of the population. In particular, as the number of agents tends to infinity, the *empirical measures* [6] converge to the measure for which this PDE's solution is the density $y(\mathbf{x}, t)$. See [41] for such a convergence analysis. Problem III.1 can be reframed in terms of equation (5) as a PDE controllability problem as follows:

Problem III.2. *Given $t_f > 0$, $y^0 : \Omega \to \bar{\mathbb{R}}_+$, and $f : \Omega \to \mathbb{R}_+$ such that $\int_\Omega y^0(\mathbf{x})d\mathbf{x} = \int_\Omega f(\mathbf{x})d\mathbf{x} = 1$, determine whether there exists a space- and time-dependent parameter $\mathbf{v} : \Omega \times [0, t_f] \to \mathbb{R}^n$ such that the solution y of the PDE* (5) *satisfies $y(\cdot, t_f) = f$.*

In models of robotic swarms, it is useful to consider *hybrid* variants of the SDE (4) to account for the fact that each robot, in addition to a continuous spatial state $\mathbf{Z}(t)$, can be associated with a discrete state $Y(t) \in \mathcal{V} = \{1, ..., N\}$ at each time t [32], [16]. The elements of $\mathcal{V}$ can correspond to different behavioral states or tasks that can be performed by a robot, such as "searching," "lifting," or "digging." In this case, the state of each agent is denoted by the pair $(\mathbf{Z}(t), Y(t)) \in \Omega \times \mathcal{V}$. Suppose that the variable $Y(t)$ evolves according to a CTMC. We define a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in which the vertex set $\mathcal{V}$ is the set of discrete states, and the edge set $\mathcal{E}$ defines the possible agent transitions between the discrete states in $\mathcal{V}$. The agents' transition rules are determined by the control parameters $u_e : [0, \infty) \to U$ for each $e \in \mathcal{E}$, also known as the *transition rates* of the associated CTMC. Here $U \subset \mathbb{R}_+$ is the set of *admissible transition rates*.

The variable $Y(t)$ evolves on the state space $\mathcal{V}$ according to the conditional probabilities

$$\mathbb{P}\left(Y(t + h) = T(e) \mid Y(t) = S(e)\right) = u_e(t)h + o(h) \tag{6}$$

for each $e \in \mathcal{E}$. Let $\mathcal{P}(\mathcal{V}) = \{\mathbf{y} \in \mathbb{R}^N_+ : \sum_v y_v = 1\}$ be the simplex of probability densities on $\mathcal{V}$. Corresponding to the CTMC is a set of ODEs that determine the time evolution of the probability densities $\mathbb{P}(Y(t) = v) = \mu_v(t) \in \mathbb{R}_+$. The forward equation is given by a system of linear ODEs,

$$\begin{aligned} \dot{\boldsymbol{\mu}}(t) &= \sum_{e \in \mathcal{E}} u_e(t)\mathbf{Q}_e\boldsymbol{\mu}(t), \quad t \in [0, \infty), \\ \boldsymbol{\mu}(0) &= \boldsymbol{\mu}^0 \in \mathcal{P}(\mathcal{V}), \end{aligned} \tag{7}$$

where $\mathbf{Q}_e$ are control matrices whose entries are given by

$$Q_e^{ij} = \begin{cases} -1 & \text{if } i = j = S(e), \\ 1 & \text{if } i = T(e),\ j = S(e), \\ 0 & \text{otherwise.} \end{cases}$$

Given these definitions, we can define a hybrid switching diffusion process $(\mathbf{Z}(t), Y(t))$ as a system of SDEs of the form

$$\begin{aligned} d\mathbf{Z}(t) &= \mathbf{v}(Y(t), \mathbf{Z}(t), t)dt + \sqrt{2\mathbf{D}}d\mathbf{W}(t) + \mathbf{n}(\mathbf{Z}(t))d\psi(t), \\ \mathbf{Z}(0) &= \mathbf{Z}_0, \end{aligned} \tag{8}$$

where $\mathbf{v} : \mathcal{V} \times \Omega \times [0, t_f] \to \mathbb{R}^n$ is the state- and time-dependent velocity vector field, and $\mathbf{D} \in \mathbb{R}^N_+$ is a vector of positive elements. Here, D_k is the diffusion parameter associated with each discrete state $k \in \mathcal{V}$. Let $\mathbf{v}_k$ denote the velocity field associated with discrete state $k \in \mathcal{V}$. Then the forward equation for this system of SDEs is given by the system of PDEs

$$\begin{aligned} (y_k)_t = D_k\Delta y_k - \nabla \cdot (\mathbf{v}_k(\mathbf{x}, t)y_k) + \mathcal{F}_k \quad & in \quad \Omega \times [0, t_f] \\ y_k(\cdot, 0) = y_k^0 \quad & in \quad \Omega \\ \mathbf{n} \cdot (\nabla y_k - \mathbf{v}_k(\mathbf{x}, t)y_k) = 0 \quad & in \quad \partial\Omega \times [0, t_f], \end{aligned} \tag{9}$$

where $k \in \mathcal{V}$ and $\mathcal{F}_k = \sum_{e \in \mathcal{E}} \sum_{j \in \mathcal{V}} u_e(t)Q_e^{kj}y_j$.

We can pose a problem for the system of SDEs (8) that is similar to the one defined in Problem III.2, with a target spatial distribution assigned to each discrete state:

Problem III.3. *Given $t_f > 0$, $\mathbf{y}^0 : \Omega \to \mathbb{R}^N_{\geq 0}$, and $\mathbf{f} : \Omega^N \to \mathbb{R}^N_+$ such that $\sum_{i \in \mathcal{V}} \int_\Omega y_i^0(\mathbf{x})d\mathbf{x} = \sum_{i \in \mathcal{V}} \int_\Omega f_i(\mathbf{x})d\mathbf{x} = 1$, determine whether there exists a set of space- and time-dependent parameters $\mathbf{v}_k : \Omega \times [0, t_f] \to \mathbb{R}^n$ and time-dependent parameters $u_e : [0, t_f] \to \mathbb{R}_+$ such that the solution $\mathbf{y}$ of the system of PDEs* (9) *satisfies $\mathbf{y}(\cdot, t_f) = \mathbf{f}$.*

It will be shown that the answer to Problem III.3 can be concluded from the answer to the controllability problem defined in Problem III.2 and the *small-time controllability* properties of the system (7). Toward this end, we recall some controllability notions from nonlinear control theory [9].

Definition III.4. *Given $U \subset \bar{\mathbb{R}}_+$ and $\boldsymbol{\mu}^0 \in \mathcal{P}(\mathcal{V})$, we define $R^U(\boldsymbol{\mu}^0, t)$ to be the set of all $\mathbf{y} \in \mathcal{P}(\mathcal{V})$ for which there exists an admissible control, $\mathbf{u} = \{u_e\}_{e \in \mathcal{E}}$, taking values in U such that there exists a trajectory of system (7) with $\boldsymbol{\mu}(0) = \boldsymbol{\mu}^0$, $\boldsymbol{\mu}(t) = \mathbf{y}$. The **reachable set from $\boldsymbol{\mu}^0$ at time** t_f is defined as*

$$R^U_{t_f}(\boldsymbol{\mu}^0) = \bigcup_{0 \leq t \leq t_f} R^U(\boldsymbol{\mu}^0, t). \tag{10}$$

Definition III.5. *The system* (7) *is said to be* **small-time locally controllable (STLC)** *from an equilibrium configuration* $\boldsymbol{\mu}^{eq} \in \mathcal{P}(\mathcal{V})$ *if the set of reachable states* $R^U_{t_f}(\boldsymbol{\mu}^{eq})$ *contains a neighborhood of* $\boldsymbol{\mu}^{eq} \in \mathcal{P}(\mathcal{V})$ *in the subspace topology of* $\mathcal{P}(\mathcal{V})$ *(as a subset of* $\bar{\mathbb{R}}^N_+$*) for every* $t_f > 0$.

Here, we have defined local controllability in terms of the subspace topology of $\mathcal{P}(\mathcal{V})$. This is because the set $\mathcal{P}(\mathcal{V})$ is invariant for the system (7) of controlled ODEs, and hence one cannot expect controllability to a full neighborhood of $\boldsymbol{\mu}^{eq}$.

Using the above definitions, we will also consider the following problem.

Problem III.6. *Given* $\boldsymbol{\mu}^{eq}$*, determine whether the system* (7) *is STLC from* $\boldsymbol{\mu}^{eq}$.

Lastly, we will consider the problem of stabilizing a target stationary distribution $\mathbf{f}^{eq}$ of the process (8) using time-independent control laws, which are more practical for implementation than time-dependent control laws:

Problem III.7. *Given* $\mathbf{f}^{eq} : \Omega \to \mathbb{R}^N_+$*, where* $\mathbf{f}^{eq} = [f_1 ... f_N]^T$*, determine whether there exist time-independent and possibly spatially-dependent parameters* $\mathbf{v}_k : \Omega \to \mathbb{R}^n$*,* $K_e : \Omega \to \mathbb{R}_+$ *such that the solution of the system*

$$\begin{aligned}(y_k)_t = D_k \Delta y_k - \nabla \cdot (\mathbf{v}_k(\mathbf{x}) y_k) + \mathcal{F}_k \quad & in \quad \Omega \times [0,\infty) \\ y_k(\cdot, 0) = y_k^0 \quad & in \quad \Omega \\ \mathbf{n} \cdot (\nabla y_k - \mathbf{v}_k(\mathbf{x}) y_k) = 0 \quad & in \quad \partial\Omega \times [0,\infty),\end{aligned}$$

where $k \in \mathcal{V}$ *and* $\mathcal{F}_k = \sum_{e \in \mathcal{E}} \sum_{j \in \mathcal{V}} K_e(\mathbf{x}) Q_e^{kj} y_j$*, satisfies* $\lim_{t\to\infty} y_k(\cdot, t) \to f_k$ *for each* $k \in \mathcal{V}$.

A similar problem can be posed for the PDE system (5). The solution to this problem for system (5), at least for smooth target stationary distributions on bounded domains, can be inferred from existing literature on *Gibbs distributions* of Fokker-Planck equations evolving on compact manifolds without boundary [39]. In this paper, we relax the smoothness requirement on the achievable stationary distributions for system (5) (see Lemma IV.4) and also consider (Euclidean) domains with possibly non-smooth boundaries $\partial\Omega$. More importantly, we show in Theorem V.5 that we can construct solutions to Problem III.7 by combining the result in Lemma IV.4 with the solution to the problem of stabilizing target stationary distributions of the finite-dimensional system (7), which we previously established in [14].

IV. Controllability Analysis

A. Controllability of PDE System (5)

In this section, we prove one of the main theorems of this paper. Specifically, we show that the PDE system (5) is controllable to a large class of sufficiently regular target probability densities. We first provide some new definitions that will be used in the subsequent analysis.

Given $a \in L^\infty(\Omega)$ such that $a \geq c$ for some positive constant c, and $\mathcal{D}(\omega_a) = H^1_a(\Omega)$, we define the *sesquilinear form* $\omega_a : \mathcal{D}(\omega_a) \times \mathcal{D}(\omega_a) \to \mathbb{C}$ as

$$\omega_a(u, v) = \int_\Omega \nabla(a(\mathbf{x})u(\mathbf{x})) \cdot \nabla(a(\mathbf{x})\bar{v}(\mathbf{x}))d\mathbf{x} \tag{11}$$

for each $u \in \mathcal{D}(\omega_a)$. We associate with the form ω_a an operator $A_a : \mathcal{D}(A_a) \to L^2_a(\Omega)$, defined as $A_a u = v$ if $\omega_a(u, \phi) = \langle v, \phi \rangle_a$ for all $\phi \in \mathcal{D}(\omega_a)$ and for all $u \in \mathcal{D}(A_a) = \{g \in \mathcal{D}(\omega_a) : \exists f \in L^2_a(\Omega) \text{ s.t. } \omega_a(g, \phi) = \langle f, \phi \rangle_a \ \forall \phi \in \mathcal{D}(\omega_a)\}$.

Similarly, given $a \in L^\infty(\Omega)$ such that $a \geq c$ for some positive constant c and $\mathcal{D}(\sigma_a) = H^1_a(\Omega)$, we define the *sesquilinear form* $\sigma_a : \mathcal{D}(\sigma_a) \times \mathcal{D}(\sigma_a) \to \mathbb{C}$ as

$$\sigma_a(u, v) = \int_\Omega \frac{1}{a(\mathbf{x})} \nabla(a(\mathbf{x})u(\mathbf{x})) \cdot \nabla(a(\mathbf{x})\bar{v}(\mathbf{x}))d\mathbf{x} \tag{12}$$

for each $u \in \mathcal{D}(\sigma_a)$. As for the form ω_a, we associate an operator $B_a : \mathcal{D}(B_a) \to L^2_a(\Omega)$ with the form σ_a. We define this operator as $B_a u = v$ if $\sigma_a(u, \phi) = \langle v, \phi \rangle_a$ for all $\phi \in \mathcal{D}(\sigma_a)$ and for all $u \in \mathcal{D}(B_a) = \{g \in \mathcal{D}(\sigma_a) : \exists f \in L^2_a(\Omega) \text{ s.t. } \sigma_a(g, \phi) = \langle f, \phi \rangle_a \ \forall \phi \in \mathcal{D}(\sigma_a)\}$.

Note that, formally, $-A_{\mathbf{1}} = -B_{\mathbf{1}}$ is the Laplacian operator $\Delta(\cdot)$ with *Neumann boundary condition* $(\mathbf{n} \cdot (\nabla \cdot\ ) = 0 \text{ in } \partial\Omega)$. For general extension domains Ω, the normal derivative might not make sense since it might not be true that $\mathcal{D}(A_{\mathbf{1}})$ is a subset of $H^2(\Omega)$ [23]. Then, the Neumann boundary condition has to be interpreted in a *weak sense*.

Using the above definitions, we derive some preliminary results on the unbounded operators $-A_a$ and $-B_a$. The semigroups generated by these operators will each play an important role in the proof of controllability of system (5).

Lemma IV.1. *The operators* $A_a : \mathcal{D}(A_a) \to L^2_a(\Omega)$ *and* $B_a : \mathcal{D}(A_a) \to L^2_a(\Omega)$ *are closed, densely-defined, and self-adjoint. Moreover, the operators have a purely discrete spectrum.*

Proof. Consider the associated form ω_a. This form is *closed*, i.e., the space $\mathcal{D}(\omega_a)$ equipped with the norm $\|\cdot\|_{\omega_a}$, given by $\|u\|_{\omega_a} = (\|u\|^2_a + \omega_a(u, u))^{1/2}$ for each $u \in \mathcal{D}(\omega_a)$, is complete. This is true due to the fact that the multiplication map $u \mapsto a \cdot u$ is an isomorphism from $H^1_a(\Omega)$ to $H^1(\Omega)$ and $H^1(\Omega)$ is a Banach space. Moreover, the space $H^1_a(\Omega)$ is dense in $L^2_a(\Omega)$. This follows from the inequality $\|au - av\|_2 \leq \|a\|_\infty \|u - v\|_2$ for each $u, v \in L^2(\Omega)$, the fact that the spaces $L^2_{\mathbf{1}}(\Omega)$ and $L^2_a(\Omega)$ are isomorphic, and the fact that the $H^1(\Omega)$ is dense in $L^2(\Omega)$. In addition, it follows from the definition of the form ω_a that ω_a is *symmetric*, meaning that $\omega_a(u, v) = \overline{\omega_a(v, u)}$ for each $u, v \in \mathcal{D}(\omega_a)$. The form ω_a is also *semibounded*, i.e., there exists $m \in \mathbb{R}$ such that $\omega_a(u, u) \geq m\|u\|^2_a$ for each $u \in \mathcal{D}(\omega_a)$. Particularly, this inequality is true for $m = 0$ since $\omega_a(u, u)$ is non-negative for all $u \in \mathcal{D}(\omega_a)$. Hence, it follows from [38][Theorem 10.7] that the operator A_a is self-adjoint. To establish the discreteness of the spectrum of A_a, we note that $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ whenever Ω is an extension domain (Definition II.2). This implies that when $H^1_a(\Omega) = \mathcal{D}(\omega_a)$ is equipped with the norm $\|\cdot\|_{\omega_a}$, then it is also compactly embedded in $L^2_a(\Omega)$. From [38][Proposition 10.6], it follows that A_a has a purely discrete spectrum.

For the operator B_a, we only check that the form σ_a is closed. The rest of the proof follows exactly the same arguments as the proof for A_a. To prove that the form σ_a is closed, we need to prove that the space $\mathcal{D}(\sigma_a)$ equipped

with the norm $\|\cdot\|_{\sigma_a}$, given by $\|u\|_{\sigma_a} = (\|u\|_a^2 + \sigma_a(u,u))^{1/2}$ for each $u \in \mathcal{D}(\sigma_a)$, is complete. Note that due to the lower bound c on a, there exist constants $k_1, k_2 > 0$ such that

$$\begin{aligned} k_1 \textstyle\int_\Omega \nabla(a(\mathbf{x})u(\mathbf{x})) \cdot \nabla(a(\mathbf{x})\bar{u}(\mathbf{x}))d\mathbf{x} \\ \leq \textstyle\int_\Omega \frac{1}{a(\mathbf{x})}\nabla(a(\mathbf{x})u(\mathbf{x})) \cdot \nabla(a(\mathbf{x})\bar{u}(\mathbf{x}))d\mathbf{x} \\ \leq k_2 \textstyle\int_\Omega \nabla(a(\mathbf{x})u(\mathbf{x})) \cdot \nabla(a(\mathbf{x})\bar{u}(\mathbf{x}))d\mathbf{x} \end{aligned} \tag{13}$$

for all $u \in H_a^1(\Omega)$. From these inequalities, it follows that $k_1\|u\|_{H_a^1} \leq \|u\|_{\sigma_a} \leq k_2\|u\|_{H_a^1}$ for all $u \in \mathcal{D}(\sigma_a) = H_a^1(\Omega)$. Hence, the form σ_a is closed. Due to the symmetry and semiboundedness of this form, it follows from [38][Theorem 10.7] that the operator is self-adjoint. Since the norm $\|\cdot\|_{\sigma_a}$ is equivalent to the norm $\|\cdot\|_{H_a^1}$, the discreteness of the spectrum of B_a again follows from [38][Proposition 10.6] due to the compact embedding of $H_a^1(\Omega)$ in $L_a^2(\Omega)$. □

Corollary IV.2. *Consider the PDE*

$$\begin{aligned} y_t = \Delta(a(\mathbf{x})y) \qquad & in \;\; \Omega \times [0,T] \\ y(\cdot,0) = y^0 \qquad & in \;\; \Omega \\ \mathbf{n} \cdot \nabla(a(\mathbf{x})y) = 0 \qquad & in \;\; \partial\Omega \times [0,T]. \end{aligned} \tag{14}$$

Let $y^0 \in L_a^2(\Omega)$. Then $-A_a$ generates a semigroup of operators $(\mathcal{T}_a^A(t))_{t\geq 0}$ such that the unique mild solution $y \in C([0,T]; L_a^2(\Omega))$ of the above PDE exists and is given by $y(\cdot,t) = \mathcal{T}_a^A(t)y^0$ for all $t \geq 0$. Additionally, the semigroup $(\mathcal{T}_a^A(t))_{t\geq 0}$ is positive. Finally, if $\|\mathcal{M}_a y^0\|_\infty \leq 1$, then $\|\mathcal{M}_a \mathcal{T}_a^A(t)y^0\|_\infty \leq 1$ for all $t \geq 0$.

Proof. First, we note that the operator $-A_a$ is *dissipative*, i.e., $\|(\lambda + A_a)u\|_a \geq \lambda\|u\|_a$ for all $\lambda > 0$ and all $u \in \mathcal{D}(A_a)$, since $\omega_a(u,u) \geq 0$ for all $u \in \mathcal{D}(\omega_a)$. Next, we note that $-A_a$ is self-adjoint, and hence the adjoint operator $-A_a^*$ is dissipative as well. It follows from a corollary of the *Lumer-Phillips theorem* [17][Corollary II.3.17] that $-A_a$ generates a semigroup of operators $(\mathcal{T}_a^A(t))_{t\geq 0}$ that solves the PDE (14) in the mild sense.

Second, we establish the positivity of the semigroup. Toward this end, we note that the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ is Lipschitz. Hence, it follows from [42][Theorem 2.1.11] that $v \in H^1(\Omega)$ implies that $|v| \in H^1(\Omega)$ whenever v is only real-valued. This implies that if $u \in \mathcal{D}(\omega_a)$, then $|\mathrm{Re}(u)| \in \mathcal{D}(\omega_a)$, where $\mathrm{Re}(\cdot)$ denotes the real component of its argument. Then the positivity of the semigroup follows from [35][Theorem 2.7].

To prove the last statement in the corollary, consider the closed convex set $C = \{u \in L^2(\Omega);\ \mathrm{Re}(u) = u,\ u(\mathbf{x}) \leq 1/a(\mathbf{x})\ a.e.\ \mathbf{x} \in \Omega\}$. The projection of a function $u \in L_a^2(\Omega)$ onto the set C can be represented by the (nonlinear) operator P, given by $Pu = \mathrm{Re}(u) \wedge 1/a = \frac{1}{2}\mathrm{Re}(u) + \frac{1}{2}|\mathrm{Re}(u) - 1/a|$. If $u \in \mathcal{D}(\omega_a)$, then it follows from the chain rule [42][Theorem 2.1.11] that $\nabla(aPu) = \frac{1}{2}\mathrm{sign}(\mathrm{Re}(au) - 1)\nabla(\mathrm{Re}(au)) + \frac{1}{2}\nabla(\mathrm{Re}(au))$. Hence, it follows that $\omega_a(Pu, Pu) \leq \omega_a(u,u)$ for all $u \in \mathcal{D}(\Omega_a)$. According to [35][Theorem 2.3], this implies that the set C is invariant under the positive semigroup $(\mathcal{T}_a^A(t))_{t\geq 0}$; therefore, we can conclude that if $\|\mathcal{M}_a y^0\|_\infty \leq 1$, then $\|\mathcal{M}_a \mathcal{T}_a^A(t)y^0\|_\infty \leq 1$ for all $t \geq 0$. □

Using the same arguments as in the proof of Corollary IV.2, we have the following result.

Corollary IV.3. *The operator $-B_a$ generates a semigroup of operators $(\mathcal{T}_a^B(t))_{t\geq 0}$ on $L_a^2(\Omega)$. If additionally $a \in W^{1,\infty}(\Omega)$ and $y^0 \in L_a^2(\Omega)$, then $y(\cdot,t) = \mathcal{T}_a^B(t)y^0$ is a mild solution of the PDE*

$$\begin{aligned} y_t = \Delta y - \nabla \cdot (\tfrac{\nabla f(\mathbf{x})}{f(\mathbf{x})}\ y) \qquad & in \;\; \Omega \times [0,T] \\ y(\cdot,0) = y^0 \qquad & in \;\; \Omega \\ \mathbf{n} \cdot (\nabla y - \tfrac{\nabla f(\mathbf{x})}{f(\mathbf{x})} y) = 0 \qquad & in \;\; \partial\Omega \times [0,T], \end{aligned} \tag{15}$$

with $f = 1/a \in W^{1,\infty}(\Omega)$. Moreover, the semigroup $(\mathcal{T}_a^B(t))_{t\geq 0}$ is positive.

When $f \in C^\infty(\bar{\Omega})$, the representation of the operator $\Delta(\cdot) - \nabla \cdot (\frac{\nabla f(\mathbf{x})}{f(\mathbf{x})}\ \cdot\ )$ in the form $\nabla \cdot (f\nabla(\frac{1}{f}\ \cdot\ ))$ is a well-known technique in the literature on Fokker-Planck equations for SDEs with drifts generated by potential functions [39]. In Corollary IV.3, however, since a is only once weakly differentiable and $\mathcal{D}(\sigma_a) = H_a^1(\Omega)$ (or equivalently, $H^1(\Omega)$), the operation Δy is not admissible unless a has additional regularity. Hence, the mild solution y should be interpreted as the weak solution of the PDE (15) when $a, f \in W^{1,\infty}(\Omega)$; i.e., it can be shown that y satisfies

$$\begin{aligned} \langle y_t, \phi\rangle_{V^*,V} = \sigma_a(u,\phi) \\ = \int_\Omega \nabla y(\mathbf{x},t) \cdot \nabla\phi(\mathbf{x})d\mathbf{x} + \int_\Omega \frac{\nabla f(\mathbf{x})}{f(\mathbf{x})} y(\mathbf{x},t) \cdot \nabla\phi(\mathbf{x})d\mathbf{x} \end{aligned} \tag{16}$$

for all $\phi \in V$ and a.e. $t \in [0,T]$, where $V = H^1(\Omega)$ and V^* is the dual space of V. Here, the second equality follows from the product rule (Theorem IV.6) and the fact that $a, f \in W^{1,\infty}(\Omega)$. Note that in the weak formulation (16), the second-order differentiability of f or y is not required. That the mild solution of a linear PDE is also a weak solution follows from standard energy estimates and weak-topology arguments. See also [4].

Next, we establish that the semigroups $(\mathcal{T}_a^A(t))_{t\geq 0}$ and $(\mathcal{T}_a^B(t))_{t\geq 0}$ are *analytic* [27]. Additionally, we will show some mass-conserving properties and long-term stability properties of these semigroups.

Lemma IV.4. *The semigroups $(\mathcal{T}_a^A(t))_{t\geq 0}$ and $(\mathcal{T}_a^B(t))_{t\geq 0}$ that are generated by the operators $-A_a$ and $-B_a$, respectively, are analytic. Additionally, these semigroups have the following mass conservation property: if $y^0 \geq 0$ and $\int_\Omega y^0(\mathbf{x})d\mathbf{x} = 1$, then $\int_\Omega (\mathcal{T}_a^A(t)y^0)(\mathbf{x})d\mathbf{x} = \int_\Omega (\mathcal{T}_a^B(t)y^0)(\mathbf{x})d\mathbf{x} = 1$ for all $t \geq 0$. Moreover, 0 is a simple eigenvalue of the operators $-A_a$ and $-B_a$. Hence, if $y^0 \geq 0$ and $\int_\Omega y^0(\mathbf{x})d\mathbf{x} = \int_\Omega f(\mathbf{x})d\mathbf{x} = 1$, then the following estimates hold:*

$$\|\mathcal{T}_a^A(t)y^0 - f\|_a \leq M_0 e^{-\lambda t}\|y^0 - f\|_a, \tag{17}$$

$$\|\mathcal{T}_a^B(t)y^0 - f\|_a \leq \tilde{M}_0 e^{-\tilde{\lambda} t}\|y^0 - f\|_a \tag{18}$$

for some positive constants $M_0, \tilde{M}_0, \lambda, \tilde{\lambda}$ and all $t \geq 0$.

Proof. The operators A_a and B_a are self-adjoint and positive semi-definite. Hence, their spectra lie in $[0,\infty)$. From this, it follows that the corresponding semigroups generated

by $-A_a$ and $-B_a$ are analytic from the definition of an analytic semigroup [27][Chapter II]. Let $\int_\Omega y^0(\mathbf{x})d\mathbf{x} = 1$ such that $y^0 \in L^2_a(\Omega)$. Then $\int_\Omega (y(\mathbf{x},t) - y^0(\mathbf{x}))d\mathbf{x} = -\int_\Omega A_a(\int_0^t y(\mathbf{x},s)ds)d\mathbf{x} = -\omega_a(\int_0^t y(\mathbf{x},s)ds, 1/a) = 0$ for all $t \geq 0$. Hence, the integral preserving property of the semigroups holds. For the exponential stability estimates (17) and (18), we note that since the domain Ω is a connected bounded extension domain, it follows from Poincaré's inequality [25][Theorem 12.23] that there exists a constant $C > 0$ such that for all $u \in H^1(\Omega)$, $\int_\Omega |u(\mathbf{x}) - u_\Omega|^2 d\mathbf{x} \leq C\int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x}$, where $u_\Omega = \frac{1}{\mu(\Omega)}\int_\Omega u(\mathbf{x})d\mathbf{x}$. This implies that 0 is a simple eigenvalue of the Neumann Laplacian operator $A_{\mathbf{1}}$. Since the operator A_a can be written as a composition of operators $A_{\mathbf{1}}\mathcal{M}_a$, where $\mathcal{M}_a$ is the multiplication map $u \mapsto au$ from $H^1_a(\Omega)$ to $H^1(\Omega)$, it follows that 0 is also a simple eigenvalue of A_a with the corresponding eigenvector $f = 1/a$. Additionally, for a given $u \in H^1_a(\Omega)$, $\omega_a(u,u) = 0$ iff $\sigma_a(u,u) = 0$ due to the assumed positive lower bound on a. Hence, it also holds that 0 is a simple eigenvalue of the operator B_a. Then we can derive the estimates (17) and (18) by applying [17][Corollary V.3.3] as follows. We note that $\mathcal{D}(A_a) \subset H^1_a(\Omega)$ and $\mathcal{D}(B_a) \subset H^1_a(\Omega)$ are compactly embedded in $L^2_a(\Omega)$, since $H^1_a(\Omega)$ is compactly embedded in $L^2_a(\Omega)$. Thus, since the semigroups $\mathcal{T}^A_a(t)$ and $\mathcal{T}^B_a(t)$ are analytic, which implies that $\mathcal{T}^A_a(t) \in \mathcal{D}(A_a)$ and $\mathcal{T}^B_a(t) \in \mathcal{D}(B_a)$ for all $y^0 \in L^2_a(\Omega)$ and all $t > 0$, these semigroups are immediately compact. Moreover, the residue of the first eigenvalue 0 in [17][Corollary V.3.3] is the projection operator onto the subspace spanned by the element $f \in L^2_a(\Omega)$, since 0 is a simple eigenvalue with the corresponding eigenvector f. □

The above result implies that if $\mathbf{v}(\cdot,t) = \nabla f/f$, then the solution of system (5) exponentially converges to f in the L^2_a-norm if f is in $W^{1,\infty}(\Omega)$ and is bounded from below by a positive constant. Hence, this choice of $\mathbf{v}(\cdot,t)$ is a possible control law for achieving exponential stabilization of desired probability densities. In the next few results, we verify some regularizing properties of the semigroups considered above, which will be critical to our controllability analysis.

Lemma IV.5. *Let $a \in L^\infty(\Omega)$ be real-valued and uniformly bounded from below by a positive constant c_1. Moreover, let $y^0 \in L^2_a(\Omega)$ such that $y^0 \geq c_2$ for some positive constant c_2. If $(\mathcal{T}^F_a(t))_{t\geq 0}$ is the semigroup generated by the operator $-A_a$ or $-B_a$, then $\mathcal{T}^F_a(t)y^0 \geq \frac{c_1c_2}{\|a\|_\infty}$ for all $t \geq 0$.*

Proof. Let $k = c_1c_2$. Then we know that $a\cdot y^0 \geq k$. Hence, we can decompose the initial condition as $y^0 = kf + (y^0 - kf)$, where $f = 1/a$. Note that $y^0 - kf$ is positive and $\mathcal{T}^B_a(t)f = f$ for all $t \geq 0$. Therefore, it follows from the positivity preserving property of the semigroup (Corollary IV.3) that $\mathcal{T}^B_a(t)y^0 \geq k/\|a\|_\infty$ for all $t \geq 0$. □

In order to prove the next result (Proposition IV.7), we will use the following well-known product rule for Sobolev functions.

Theorem IV.6. *(Product Rule) Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Suppose that $u \in H^1(\Omega)$ and $v \in W^{1,\infty}(\Omega)$. Then $u \cdot v \in H^1(\Omega)$, and the weak derivatives of the product $u\cdot v$ are given by $(uv)_{x_i} = u_{x_i}v + v_{x_i}u$ for each $i \in \{1,...,n\}$.*

Proposition IV.7. *Let $a \in W^{1,\infty}(\Omega)$ and uniformly bounded from below by a positive constant c. Then $\mathcal{D}(A_a) = \mathcal{D}(B_a)$.*

Proof. Let $u \in \mathcal{D}(B_a)$. Then using the product rule (Theorem IV.6), we have that $\omega_a(u, \frac{\phi}{a}) = \langle B_a u, \phi\rangle_a - \langle \frac{1}{a^2}\nabla a \cdot \nabla(au), \phi\rangle_a$ for all $\phi \in H^1_a(\Omega)$. Since a is in $W^{1,\infty}(\Omega)$ and is bounded from below by a positive constant, $H^1(\Omega) = H^1_a(\Omega)$. Hence, $\phi \in H^1_a(\Omega)$ implies that $a \cdot u$, $\frac{\phi}{a} \in H^1(\Omega)$ due to the product rule (Theorem IV.6). Therefore, we can conclude that $\omega_a(u,\phi) = \langle a \cdot B_a u, \phi\rangle_a - \langle \frac{1}{a}\nabla a \cdot \nabla(au), \phi\rangle_a$ for all $\phi \in H^1_a(\Omega)$. Hence, $u \in \mathcal{D}(B_a)$ implies $u \in \mathcal{D}(A_a)$. To establish that $u \in \mathcal{D}(A_a)$ implies $u \in \mathcal{D}(B_a)$, we can use a similar argument: if $u \in \mathcal{D}(A_a)$, then $\sigma_a(u, a\phi) = \langle A_a u, \phi\rangle_a + \langle \frac{1}{a}\nabla a \cdot \nabla(au), \phi\rangle_a$ for all $\phi \in H^1_a(\Omega)$. □

This lemma will play an important role in the theorem on controllability, Theorem IV.13. It will be used to conclude that solutions of the parabolic systems (14) and (15) have bounded gradients for each $t > 0$, provided that the boundary of the domain Ω is regular enough. This will enable us to prove later on that the control inputs constructed to prove controllability are bounded.

Lemma IV.8. *Let $a \in W^{1,\infty}(\Omega)$. Let Ω be a domain that is either $C^{1,1}$ or convex. Then there exists $m \in \mathbb{Z}_+$ large enough such that, for some $C_m > 0$, $\|\nabla(a(\mathbf{x})u)\|_\infty \leq C_m(\|(\mathbb{I} + A_a)^m u\|_a)$ for all $u \in \mathcal{D}(A^m_a)$. Similarly, there exists $m' \in \mathbb{Z}_+$ large enough such that, for some $C_{m'} > 0$, $\|\nabla(a(\mathbf{x})u)\|_\infty \leq C_{m'}(\|(\mathbb{I} + B_a)^{m'} u\|_a)$ for all $u \in \mathcal{D}(B^{m'}_a)$.*

Proof. First, we consider the case where Ω is a $C^{1,1}$ domain. Let $W^{2,p}(\Omega)$ be the set of elements in $L^p(\Omega)$ with second-order weak derivatives in $L^p(\Omega)$. Then we know that for the Neumann problem with $a = \mathbf{1}$,

$$-\Delta u + a_0 u = f \ \text{ in } \Omega, \quad \mathbf{n}\cdot\nabla u = 0 \ \text{ in } \partial\Omega \tag{19}$$

has solutions $u \in W^{2,p}(\Omega)$ if $f \in L^p(\Omega)$ whenever $1 < p < \infty$, $a_0 \in L^\infty(\Omega)$ [22][Theorem 2.4.2.7]) and $a_0 \geq \beta$ for some $\beta > 0$. These solutions have bounds

$$\|u\|_{W^{2,p}} \leq C_p\|f\|_{L^p} \tag{20}$$

for some constant $C_p > 0$. Note that $u \in L^p(\Omega)$ implies that $\mathcal{M}_a u \in L^p(\Omega)$ for each $2 \leq p \leq \infty$. Suppose that $n > 2$, where we recall that n is the dimension of the Euclidean space $\mathbb{R}^n$ of which Ω is a subset. Note that Ω is an extension domain (Definition II.2). Then from the $W^{2,p}$ *regularity estimate* (20) of equation (19) and from the embedding theorem [25][Corollary 11.9], it follows that $f \in L^2(\Omega)$ implies $(A_a + \mathbb{I})^{-m} f = (A_{\mathbf{1}}\mathcal{M}_a + \mathbb{I})^{-m} f = \left((A_{\mathbf{1}} + \mathcal{M}_a^{-1})\mathcal{M}_a\right)^{-m} f \in L^q(\Omega)$ for some $q \geq n$ for $m \in \mathbb{Z}_+$ large enough. For the general case $n \geq 1$, it follows from the embedding theorem [25][Theorem 11.23] that $f \in L^2(\Omega)$ implies $(A_a + \mathbb{I})^{-m} f \in L^q(\Omega)$ for any desired $n \leq q < \infty$, provided $m \in \mathbb{Z}_+$ for m large enough. Since $a \in W^{1,\infty}(\Omega)$, it follows from the $W^{2,p}$ regularity estimate (20) of equation (19) and *Morrey's inequality* [25][Theorem 11.34] that if $f \in L^2(\Omega)$, $(A_a + \mathbb{I})^{-m} f = u \in L^q(\Omega)$, and $m \in \mathbb{Z}_+$ is

large enough, then $\|\nabla u\|_\infty \leq C_\infty \|f\|_2$, where $C_\infty > 0$ is independent of f.

A similar argument can be used when Ω is convex. However, it is not clear if the $W^{2,p}$ regularity estimate (20) holds for general convex domains. On the other hand, it can be established that the L^p *regularity estimate* of the PDE (19) holds for such domains. In particular, it follows from [3][Corollary 6.3.3] and the embedding theorems [25][Corollary 11.9, Theorem 11.23] that for any $1 < p \leq \infty$, there exists $m \in \mathbb{Z}_+$ large enough such that $(A_a + \mathbb{I})^{-m}$ is a bounded operator from $L^2(\Omega)$ to $L^p(\Omega)$. This last statement uses only the extension property of the domain Ω, which is not required to be convex for the statement to hold true; the convexity of Ω is required mainly to derive the bounds on the gradient of the solution u. For this derivation, we use a result from [29]. Since $a \in W^{1,\infty}(\Omega)$, it follows from the theorem [29][Theorem] that there exists a constant $C'_\infty > 0$ such that if $f \in L^2(\Omega)$, $(A_a + \mathbb{I})^{-m} f = u$, and $m \in \mathbb{Z}_+$ is large enough, then $\|\nabla u\|_\infty \leq C'_\infty \|f\|_2$, where C'_∞ is independent of f. In this last statement, the theorem [29][Theorem] can be applied to derive the gradient bounds due to the fact that $(A_a + \mathbb{I})^{-m+1} f \in L^q(\Omega)$ for some $q > n$ for $m \in \mathbb{Z}_+$ large enough.

The inequality for the operator B_a can be derived using exactly the same approach as for A_a. Hence, we only point out the key results needed. Particularly, for a $C^{1,1}$ domain, the $W^{2,p}$ regularity estimate (20) also holds for the equation $-\nabla \cdot (\frac{1}{a(\mathbf{x})} \nabla u) + a_0 u = f$ from [22][Theorem 2.4.2.7]. For the case where Ω is convex, the $W^{1,p}$ regularity estimate has been proved in [21][Theorem 1.3] for solutions of elliptic operators in divergence form on convex domains. Since $a \in W^{1,\infty}(\Omega)$, using the product rule (Theorem IV.6), the gradient bounds of the Neumann Laplacian [29][Theorem] also give the desired gradient bounds of the operator $\nabla \cdot (\frac{1}{a(\mathbf{x})} \nabla \cdot)$. □

Lemma IV.9. *Let Ω be a domain that is either $C^{1,1}$ or convex. Let $y^0 \in \mathcal{D}(A_a^m)$ for some $m \in \mathbb{Z}_+$. Then the mild solution, $y \in C([0,\infty); L^2_a(\Omega))$, of the PDE* (14) *satisfies $y(\cdot, t) \in \mathcal{D}(A_a^m)$ for each $t \in [0,\infty)$. Moreover, the following estimates hold for some positive constants M_m and λ:*

$$\|(\mathbb{I} + A_a)^m (y^0 - f)\|_a \;\leq\; M_m e^{-\lambda t}. \tag{21}$$

Proof. We are given that $y^0 \in \mathcal{D}(A_a^m)$. Since the semigroup $(\mathcal{T}_a^A(t))_{t\geq 0}$ and its generator $-A_a$ commute, we know that $\|(\mathbb{I} + A_a)^m (\mathcal{T}_a^A(t))(y^0 - f)\|_a = \|\mathcal{T}_a^A(t)(\mathbb{I} + A_a)^m (y^0 - f)\|_a \leq M_0 e^{-\lambda t} \|(\mathbb{I} + A_a)^m (y^0 - f)\|_a$ for some positive constants M_0 and λ. □

Since controllability will first be proved in Lemma IV.11 under the assumption that the initial condition is bounded from below by a positive constant, the following lemma will be used to relax this assumption in Theorem IV.13.

Lemma IV.10. *Let Ω be a domain that is either $C^{1,1}$ or convex and that satisfies the chain condition (see Definition II.3). Let $y^0 \in L^2(\Omega)$ be such that $y^0 \geq 0$. Let $y \in C([0,T]; L^2(\Omega))$ be the unique mild solution of the PDE* (14)*. Then for all $t \in (0,\infty)$, there exists a positive constant, $c_t > 0$, such that $y(\cdot, t) \geq c_t$.*

Proof. Consider the heat equation with Neumann boundary condition, that is, the PDE (5) with $\mathbf{v} \equiv \mathbf{0}$. The solution y of this PDE can be represented using the *Neumann heat kernel* K. That is, there exists a measurable map $K : (0,\infty) \times \Omega^2 \to [0,\infty)$ such that the mild solution y can be constructed using the relation $y(\mathbf{x}, t) = \int_\Omega K(t, \mathbf{x}, \mathbf{z}) y^0(\mathbf{z}) d\mathbf{z}$ for each $t \in (0,\infty)$ and almost every $\mathbf{x} \in \Omega$. From [12][Theorem 3.1] (for $C^{1,1}$ domains) and [26][Corollary 2.1] (for convex domains), for some $C > 0$, we know that $K(t, \mathbf{x}, \mathbf{z}) \geq \frac{C}{(4\pi t)^{1/2}} \exp(\frac{-|\mathbf{x}-\mathbf{z}|^2}{4t})$ for each $t > 0$ and almost every $\mathbf{x}, \mathbf{z} \in \Omega$. From this, the lower bound on $y(\cdot, t)$ follows. □

In the following lemma, we establish a controllability result under the assumption that the initial condition is sufficiently regular and the final time t_f is equal to the infinite summation $\sum_{k=1}^\infty \frac{1}{k^2}$.

Lemma IV.11. *Let $y^0 \in \mathcal{D}(A_a^m)$ for some $m \in \mathbb{Z}_+$ such that $y^0 \geq c_1$ for some positive constant c_1. Suppose that $f \in W^{1,\infty}(\Omega)$ such that $f \geq c_2$ for some positive constant c_2 and $\int_\Omega f(\mathbf{x}) d\mathbf{x} = \int_\Omega y^0(\mathbf{x}) d\mathbf{x}$. Assume that the final time is specified as $t_f = \sum_{k=1}^\infty \frac{1}{k^2}$. Define the vector field $\mathbf{v}$ in the PDE* (5) *by*

$$\mathbf{v}(\cdot, t) = \frac{\nabla y}{y} - \alpha j \frac{\nabla (ay)}{y} \tag{22}$$

for some $\alpha > 0$, $j \in \mathbb{Z}_+$, where $a = 1/f$ whenever $t \in [\sum_{k=1}^{j-1} \frac{1}{k^2}, \sum_{k=1}^{j} \frac{1}{k^2})$. Here, we define $\sum_{k=1}^{j} \frac{1}{k^2} = 0$ if $j = 0$.

If Ω is a domain that is $C^{1,1}$ or convex, then there exist $m' \in \mathbb{Z}_+$ and $\alpha > 0$ large enough and independent of m such that if $m \geq m'$, then $\mathbf{v} \in L^\infty([0, t_f]; L^\infty(\Omega)^n)$ and $y(\cdot, t_f) = f$.

Proof. Substituting $\mathbf{v}(\cdot, t) = \frac{\nabla y}{y} - \alpha j \frac{\nabla (ay)}{y}$ whenever $t \in [\sum_{k=1}^{j-1} \frac{1}{k^2}, \sum_{k=1}^{j} \frac{1}{k^2})$ in the PDE (5), it can be seen that if the solution of this PDE exists, then it can be constructed from mild solutions of the *closed-loop* PDE

$$\begin{aligned} \tilde{y}_t &= \alpha j \Delta (a(\mathbf{x}) \tilde{y}) && \text{in } \ \Omega \times [0, \frac{1}{j^2}) \\ \tilde{y}(\cdot, 0) = \tilde{y}^0 &= y(\cdot, \textstyle\sum_{k=1}^{j-1} \frac{1}{k^2}) && \text{in } \ \Omega \\ \mathbf{n} \cdot \nabla (a\tilde{y}) &= 0 && \text{in } \ [0, \frac{1}{j^2}). \end{aligned} \tag{23}$$

In particular, if $t \in [\sum_{k=1}^{j-1} \frac{1}{k^2}, \sum_{k=1}^{j} \frac{1}{k^2})$, then $t = i + \sum_{k=1}^{j-1} \frac{1}{k^2}$ for some $i \in [0, \frac{1}{j^2})$. Hence, we obtain the relation $y(\cdot, t) = y(\cdot, \sum_{k=1}^{j-1} \frac{1}{k^2} + i) = \tilde{y}(\cdot, i)$. This implies that $y(\cdot, \sum_{k=1}^{j-1} \frac{1}{k^2} + i) = \tilde{y}(\cdot, i)$ for each $i \in [0, \frac{1}{j^2})$ and each $m \in \mathbb{Z}_+$. Since y^0 is uniformly bounded from below by a positive constant, $\tilde{y}$ is also uniformly bounded from below according to Lemma IV.10. Moreover, since $a \in W^{1,\infty}(\Omega)$, it follows that $\mathcal{D}(A_a) \subset H^1_a(\Omega)$. Due to the regularity assumption on the initial condition, we know that $y(\cdot, t) \in \mathcal{D}(A_a)$ for all $t \in [0, t_f)$. Hence, the velocity field $\mathbf{v}$ is well-defined for all $t \in [0, t_f)$, i.e., it is at least square-integrable at almost every time t.

Noting that $-\beta A_a$ generates the rescaled semigroup $(\mathcal{T}_a^A(\beta t))_{t\geq 0}$, and applying Lemma IV.4, it follows that $\|y(\cdot, \sum_{k=1}^{j} \frac{1}{k^2}) - f\|_a = \|\mathcal{T}_a^A(\alpha k/k^2) ... \mathcal{T}_a^A(\alpha) y^0 - f\|_a =$

$\|\mathcal{T}_a^A(\alpha\sum_{k=1}^j \frac{1}{k})y_0 - f\|_a \leq M_0 e^{-\alpha\lambda\sum_{k=1}^j \frac{1}{k}}$ for each $j \in \mathbb{Z}_+$, for some positive constants M_0 and λ independent of j. Since the summation $\sum_{k=1}^j \frac{1}{k}$ is diverging, we have that $y(\cdot, t_f) = f$ if the solution is defined over the interval $[0, t_f]$. Since y is continuous on $[0, t_f)$ and uniformly bounded, it follows that y is in $C([0, t_f); L_a^2(\Omega))$ and can be extended to a unique mild solution $y \in C([0, t_f]; L_a^2(\Omega))$ defined over the time interval $[0, t_f]$.

It is additionally required to prove the existence of $m' \in \mathbb{Z}_+$ and $\alpha > 0$ such that, if $m \geq m'$, then $\mathbf{v} \in L^\infty([0, t_f]; L^\infty(\Omega)^n)$. First, we derive bounds on the term $1/y(\cdot, t)$. Due to the lower bound on the initial condition y^0, and noting that $y(\cdot, t) = \mathcal{T}_a^A(\tilde{t})y^0$ for some $\tilde{t} \in [0, \infty)$ depending on $t \in [0, t_f)$, it follows from Lemma IV.5 that there exists a positive constant d such that

$$y(\cdot, t) \geq d \tag{24}$$

for all $t \in [0, t_f)$. This gives us the uniform upper bound $1/d$ on the term $1/y(\cdot, t)$. Next, we consider the term $\alpha\nabla(a(\mathbf{x})y(\cdot, t))$. We note that $y_0 \in \mathcal{D}(A_a^j)$. Hence, we can apply the estimate in Lemma IV.9 to obtain $\|(\mathbb{I} + A_a)^m(y(\cdot, \sum_{k=1}^m \frac{1}{k^2}) - f)\|_a \leq \tilde{M} e^{-\alpha\lambda\sum_{k=1}^j \frac{1}{k}}$ for some positive constants $\tilde{M}$ and λ. From Lemma IV.8, it follows that when Ω is a domain that is $C^{1,1}$ or convex, there exist $C > 0$ and $m' \in \mathbb{Z}_+$ depending only on a such that, if $m \geq m'$, then

$$\|\alpha j\nabla(a(\mathbf{x})y)(\cdot, \sum_{k=1}^j \frac{1}{k^2})\|_\infty \leq C\alpha j\tilde{M} e^{-\alpha\lambda\sum_{k=1}^j \frac{1}{k}} \tag{25}$$

for some positive constants $\tilde{M}$ and λ. The right-hand side of the estimate (25) is not uniformly bounded for arbitrary $\alpha > 0$ due to its dependence on j. However, we note that $\lim_{j\to\infty} -\ln j + \sum_{k=1}^j \frac{1}{k} = \gamma$, where $\gamma > 0$ is the *Euler-Mascheroni* constant [18][Section 1.5]. Therefore, by setting $\alpha \geq 1/\lambda$, the right-hand side becomes uniformly bounded for all $j \in \mathbb{Z}_+$. Since $a \in W^{1,\infty}(\Omega)$, it follows from the product rule and the estimate (25) that

$$\|\nabla y(\cdot, t)\|_\infty \leq C_2 \tag{26}$$

for some positive constant C_2 and for all $t \in [0, t_f)$.

From the estimates (24)-(26), it follows that if $\alpha > 0$ is large enough, then $\mathbf{v} \in L^\infty([0, t_f]; L^\infty(\Omega)^n)$ and $y(\cdot, t_f) = f$. This concludes the proof for the case when the domain Ω is $C^{1,1}$ or convex. □

Note that any control law of the form $\mathbf{v}(\cdot, t) = \frac{\nabla y}{y} - \alpha m^\beta \frac{\nabla(ay)}{y}$ for numerous other values of β and α will also achieve the desired controllability objective, due to the fact that an exponential function of a variable grows faster than a polynomial function as the variable tends to infinity. Additionally, we could replace the parameter m with a continuous function $m(t)$ such that $\int_0^T m(\tau)d\tau = \infty$.

The following corollary follows from Lemma IV.11 using a straightforward scaling argument.

Corollary IV.12. *Let $y^0 \in \mathcal{D}(A_a^m)$ be such that $y^0 \geq c_1$ for some positive constant c_1 and $m \in \mathbb{Z}_+$. Let Ω be a domain that is either $C^{1,1}$ or convex. Suppose that $f \in W^{1,\infty}(\Omega)$ such that $f \geq c_2$ for some positive constant c_2, $\int_\Omega f(\mathbf{x})d\mathbf{x} = \int_\Omega y^0(\mathbf{x})d\mathbf{x}$, and $a = 1/f$. Let $t_f > 0$ be the final time. Then if m is large enough there exists $\mathbf{v} \in L^\infty([0, t_f]; L^\infty(\Omega)^n)$ such that the mild solution y of the PDE (5) satisfies $y(\cdot, t_f) = f$.*

Now, we are ready to state and prove our main theorem, where we relax the assumptions on the initial condition y^0 made in Corollary IV.12. However, we will need to impose the additional constraint that Ω should satisfy the chain condition.

Theorem IV.13. *Let Ω be a domain that is either $C^{1,1}$ or convex and that satisfies the chain condition. Let $y^0 \in L^2(\Omega)$ be such that $y^0 \geq 0$ and $\int_\Omega y^0(\mathbf{x})d\mathbf{x} = 1$. Suppose that $f \in W^{1,\infty}(\Omega)$ such that $f \geq c$ for some positive constant c, $\int_\Omega f(\mathbf{x})d\mathbf{x} = 1$. Let $t_f > 0$ be the final time. Then there exists $\mathbf{v} \in L^\infty([0, t_f]; L^\infty(\Omega)^n)$ such that the unique mild solution y of the PDE (5) satisfies $y(\cdot, t_f) = f$.*

Proof. Set $\mathbf{v}(\cdot, t) = \mathbf{0}$ in the PDE (5) for each $t \in [0, \epsilon/3]$, where $\epsilon \in (0, t_f)$ is small enough. Then this PDE is the heat equation with Neumann boundary condition. From Lemma IV.10, it follows that the solution y satisfies $y(\cdot, \epsilon/2) \geq c$ for some positive constant c. For each $t \in (\epsilon/3, 2\epsilon/3]$, let $\mathbf{v}(\cdot, t) = \frac{\nabla f}{f}$. Then the mild solution of the PDE is given by the semigroup $(\mathcal{T}_a^B(t))_{t\geq 0}$, where $a = 1/f$. From Lemma IV.4, the semigroup $(\mathcal{T}_a^B(t))_{t\geq 0}$ is analytic. Hence, from regularizing properties of analytic semigroups [27][Theorem 2.1.1], it follows that $y(\cdot, \epsilon) \in \mathcal{D}(B_a^j)$ for each $j \in \mathbb{Z}_+$. From Lemma IV.8, this implies that $\|B_a y(\cdot, 2\epsilon/3)\|_\infty < \infty$. Due to the product rule (Theorem IV.6), Proposition IV.7, and Lemma IV.8, this inequality implies that $\|A_a y(\cdot, 2\epsilon/3)\|_\infty < c$ for some $c > 0$. Let $\mathbf{v}(\cdot, t) = \frac{\nabla y}{y} - \frac{\nabla(ay)}{y}$ for $t \in [2\epsilon/3, \epsilon]$. Then from the last statement of Corollary IV.2 and the fact that the operator $-A_a$ commutes with the semigroup it generates, it follows that $\|\mathcal{M}_a A_a y(\cdot, t)\|_\infty = \|\mathcal{M}_a \mathcal{T}_a^A(t - 2\epsilon/3)_a A_a y(\cdot, 2\epsilon/3)\|_\infty < c'$ for some $c' > 0$ and for all $t \in (2\epsilon/3, \epsilon]$. Since $a \in W^{1,\infty}(\Omega)$, we can apply the result in Proposition IV.7 and the gradient estimates of the Neumann Laplacian in [22][Theorem 2.4.2.7] and [29][Theorem]. Taken together, all of these observations imply that $\|\nabla y(\cdot, t)\|_\infty < k$ for some $k > 0$ and all $t \in (2\epsilon/3, \epsilon]$. From Lemma IV.5, it also follows that y is uniformly bounded from below, and hence $\mathbf{v}(\cdot, t)$ is essentially bounded for all $t \in [0, \epsilon]$. Lastly, due to the analyticity of the semigroup $(\mathcal{T}_a^A(t))_{t\geq 0}$, $y(\cdot, \epsilon) \in \mathcal{D}(A_a^j)$ for each $j \in \mathbb{Z}_+$. Then the result follows from Corollary IV.12. □

In the following theorem, we note that system (5) has stronger controllability properties than Theorem IV.13 describes: this system is *path controllable* if the path is confined to a subset of $L^2(\Omega)$ that is regular enough. This should not be very surprising due to the large dimensionality of its control inputs as compared to the choice of controls in classical PDE control problems, where control inputs are typically localized on a small subset of the interior or boundary of the domain. We restrict the path to the space $W^{2,\infty}(\Omega)$ for simplicity.

Theorem IV.14. *Let Ω be a domain that is either $C^{1,1}$ or convex, and t_f be given. Suppose that $\gamma \in C^1([0, 1]; W^{2,\infty}(\Omega))$ such that $\gamma(t) \geq c$ for some positive constant c and for all*

$t \in [0, t_f]$. *Additionally, suppose that* $\int_\Omega \gamma(\mathbf{x}, t) d\mathbf{x} = 1$ *for all* $t \in [0, t_f]$. *Then there exists* $\mathbf{v} \in L^\infty([0,1]; L^\infty(\Omega)^n)$ *such that a solution of the PDE* (5) *satisfies* $y(\cdot, t) = \gamma(t)$ *for all* $t \in [0, t_f]$, *provided that* $\gamma(0) = y^0$.

Proof. Fix $t \in [0, t_f]$. Consider the solution $\phi(t) \in L^2(\Omega)$ of the Poisson equation in weak form,

$$\omega_{\mathbf{1}}(\phi(t), \mu) = \langle \frac{\partial \gamma}{\partial t}(t), \mu \rangle \quad \text{for all } \mu \in H^1(\Omega), \tag{27}$$

where $\mathbf{1}$ is the function taking value 1 everywhere on Ω. Note that since $\int_\Omega \gamma(\mathbf{x}, t) d\mathbf{x} = 1$ for all $t \in [0, t_f]$, we have that $\int_\Omega \frac{\partial \gamma}{\partial t}(\mathbf{x}, t) d\mathbf{x} = 0$ for each $t \in [0, t_f]$, and therefore the Poisson equation has a unique solution for each $t \in [0, t_f]$. Then it follows from [22][Theorem 2.4.2.7] and Morrey's inequality [25][Theorem 11.34] (when Ω is a $C^{1,1}$ domain) and [29][Theorem] (when Ω is convex) that there exists a constant C such that $\|\nabla \phi(t)\|_\infty \leq C \|\nabla(\partial \gamma(t)/\partial t)\|_2 \leq C \|\nabla(\partial \gamma(t)/\partial t)\|_\infty$ for each $t \in [0, t_f]$. Then setting $\mathbf{v}(\cdot, t) = \frac{\nabla \gamma(t)}{\gamma(t)} - \frac{\nabla \phi(t)}{\gamma(t)}$ for each $t \in [0, t_f]$ gives us the desired controllability result. □

Note that this approach to prove path controllability using the Poisson equation coincides with the approach used by Otto in [34] to define a formal Riemannian metric on the space of probability densities, when the domain Ω is $\mathbb{R}^n$.

B. Controllability of ODE System (7)

In this section, we investigate the controllability properties of the system (7).

Proposition IV.15. *If the graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is not strongly connected, then the system* (7) *is not locally controllable.*

Proof. Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is not strongly connected. Then there exist vertices v_1, $v_2 \in \mathcal{V}$ such that there does not exist a path in $\mathcal{E}$ from v_2 to v_1. Let $\mathcal{V}_1$ be the (non-empty) subset of vertices $v \in \mathcal{V}$ such that $v = v_1$ or there exists a path in $\mathcal{E}$ from v to v_1. Analogously, let $\mathcal{V}_2$ be the (non-empty) subset of vertices $v \in \mathcal{V}$ such that $v = v_2$ or there exists a path in $\mathcal{E}$ from v_2 to v. Since there does not exist a path in $\mathcal{E}$ from v_2 to v_1, it is clear that $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint. By construction, the set $\mathcal{V}_1$ is the largest backward invariant subset of vertices containing v_1, and $\mathcal{V}_2$ is the largest forward invariant subset of vertices containing v_2. Since no edge enters $\mathcal{V}_1$ from $\mathcal{V} - \mathcal{V}_1$, and no edge leaves $\mathcal{V}_2$, the total mass in $\mathcal{V}_1$ can only decrease, and the total mass in $\mathcal{V}_2$ can only increase. Then the output function $\varphi \colon \mathcal{P}(\mathcal{V}) \mapsto \mathbb{R}$ defined by

$$\varphi(\boldsymbol{\mu}) = \sum_{v \in \mathcal{V}_2} \mu_v - \sum_{v \in \mathcal{V}_1} \mu_v \tag{28}$$

is nondecreasing along every solution curve of the system (7), which therefore is not locally controllable. □

The following proposition will be used to address Problem III.6.

Proposition IV.16. *If the graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is strongly connected, then the system* (7) *is STLC from every point in* $\operatorname{int}(\mathcal{P}(\mathcal{V}))$.

Before proving the proposition, it is helpful to take a closer look at the relations in the Lie algebra of the control vector fields, and the corresponding product on the semigroup generated by their exponentials.

Suppose that $e = (i, j), e' = (k, \ell) \in \mathcal{E}$ are two edges. If $\{i, j\}, \{k, \ell\} \subseteq \mathcal{V}$ are disjoint, then the control matrices $\mathbf{Q}_{(i,j)}$ and $\mathbf{Q}_{(k,\ell)}$ commute, and hence so do their exponentials. If $k = j$ and $\ell \neq j$, then $\mathbf{Q}_{(i,j)} \mathbf{Q}_{(j,\ell)} = \mathbf{0}$, and the commutator evaluates to

$$[\mathbf{Q}_{(j,\ell)}, \mathbf{Q}_{(i,j)}] = \mathbf{Q}_{(j,\ell)} \mathbf{Q}_{(i,j)} = \mathbf{Q}_{(i,j)} - \mathbf{Q}_{(i,\ell)}. \tag{29}$$

From this, we can conclude that if the graph is strongly connected, then the Lie algebra spanned by the control vector fields $f_e \colon \boldsymbol{\mu} \mapsto \mathbf{Q}_e \boldsymbol{\mu}$ spans the tangent space $T_{\boldsymbol{\mu}}(\mathcal{P}(\mathcal{V}))$ at every point $\boldsymbol{\mu} \in \operatorname{int}(\mathcal{P}(\mathcal{V}))$. However, since in our case $\mathbf{0}$ is not an interior point of the convex hull of admissible control values $\mathbf{u} \in [0, \infty)^M$, classical results on STLC do not apply directly.

For any edge $e = (i, j) \in \mathcal{E}$, the exponential of the control matrix $\mathbf{Q}_e$ is a stochastic matrix with entries given by

$$(\exp t\mathbf{Q}_e)_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \neq S(e) \\ e^{-t} & \text{if } k = \ell = S(e) \\ 1 - e^{-t} & \text{if } k = T(e) \text{ and } \ell = S(e) \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

Rather than writing out a general formula for the corresponding product on the group for arbitrary edges $(i, j), (j, \ell) \in \mathcal{E}$, we only state the product for the special case of $\mathcal{V} = \{1, 2, 3\}$ and edges $e = (1, 2)$ and $e' = (2, 3)$ as an illustration:

$$e^{t\mathbf{Q}_{e'}} e^{s\mathbf{Q}_e} = \begin{pmatrix} e^{-s} & 0 & 0 \\ e^{-t}(1 - e^{-s}) & e^{-t} & 0 \\ (1 - e^{-t})(1 - e^{-s}) & 1 - e^{-t} & 1 \end{pmatrix}. \tag{31}$$

Proof. (of Proposition IV.16). Suppose that the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is strongly connected. Fix an arbitrary point $\boldsymbol{\mu}^0 \in \operatorname{int}(\mathcal{P}(\mathcal{V}))$. Then there exists $\rho > 0$ such that each coordinate $\mu_i^0 > 2\rho$. Let $\Delta\boldsymbol{\mu} \in [-\rho/M, \rho/M]^M$ be an arbitrary but fixed vector such that $\sum_{v \in \mathcal{V}} \Delta\mu_v = 0$. Let the final time $T > 0$ be arbitrary but fixed. We explicitly construct a piecewise constant control $\mathbf{u} \colon [0, T] \mapsto [0, \infty)^{N_\mathcal{E}}$ that steers the system (7) from $\boldsymbol{\mu}^0$ at time 0 to $\boldsymbol{\mu}^0 + \Delta\boldsymbol{\mu}$ at time T.

Let $v_0 \in \mathcal{V}$ be arbitrary but fixed. As a consequence of Proposition IV.15, there exists a path $\gamma = (e_1, \ldots, e_s)$ of edges in $\mathcal{E}$ that connects $v_0 = S(e_1)$ back to $T(e_s) = v_0$ and which *visits* every vertex $v \in \mathcal{V}$ at least once. For $1 \leq i \leq s$, let $v_i = T(e_i)$. Let $\Delta t = T/s$. Define the finite sequence $\{\delta_i\}_{i=1}^s \in \{0, 1\}^s$ by $\delta_i = 1$ if for all $i < j < s$, $v_j \neq v_i$, i.e. the edge $e_i \in \gamma$ is the last edge whose source is $S(e_i) = v_i$. This sequence ensures that a control variation in the direction of μ_v is only taken along the last edge that starts at v; in other words, that mass is removed from the vertex v only during the last time interval when the path visits this vertex. Finally, define a finite sequence $\{\sigma_i\}_{i=0}^s \in [-\rho, \rho]^s$ that keeps track of the accumulated control variations, where

$$\sigma_0 = 0, \qquad \sigma_i = \sum_{j=1}^{i} \delta_j \Delta\mu_{v_{j-1}}, \qquad 1 < i \leq s. \tag{32}$$

Note that if the path γ is a Hamiltonian cycle, then $s = M$, $\delta_i = 1$ for all $i = 1, ..., s$, and $\sigma_i = \sum_{j=1}^{i} \Delta\mu_{v_{j-1}}$, which simplifies the formula (33) below.

To distinguish between the two cases where $S(e_i) = v_i = v_0$ and $S(e_i) = v_i \neq v_0$, we introduce the vector $\mathbf{y}^0 \in \mathbb{R}^M$ by setting $y^0_{v_0} = \mu_{v_0} - \rho$ and $y^0_{v_i} = \mu_{v_i}$ if $v_i \neq v_0$. Consider the piecewise constant control $\mathbf{u}\colon [0, T] \mapsto [0, \infty)^{N\varepsilon}$ that is defined on each interval $t \in [(i-1)\Delta t, i\Delta t)$, $1 \leq i < s$, as

$$u_{e_i}(t) = -\frac{1}{\Delta t} \ln \left(1 - \frac{\rho - \sigma_i}{y^0_{v_i} + \rho - \sigma_{i-1}} \right) \tag{33}$$

and $u_e(t) \equiv 0$ for all $e \neq e_i$.

The key idea in this construction is that the much simpler control obtained by setting $\Delta\boldsymbol{\mu} = \mathbf{0}$ in definition (32) successfully moves a mass $\rho > 0$ from v_0 along the path γ and back to v_0. It is critical that the component u_{e_i} of the control be strictly positive on the i^{th} interval of time, which enables the application of classical signed control variations to this component on the interval. The explicit introduction of the nonzero $\Delta\boldsymbol{\mu}$ then allows the following endpoint map to be calculated explicitly:

$$\prod_{1 \leq i \leq s}^{\longleftarrow} \exp\left(\Delta t u_{e_i} \mathbf{Q}_{e_i}\right) \cdot \boldsymbol{\mu}^0 = \boldsymbol{\mu}^0 + \Delta\boldsymbol{\mu}, \tag{34}$$

resulting in equation (33) for the control.

In the first time interval $[0, \Delta t]$, note that $\sigma_0 = 0$ and $y_{v_0} = \mu_{v_0} - \rho$, and hence the only nonzero and non-unity entries of the matrix exponential take the simpler forms

$$\begin{aligned}
\left(\exp(\Delta t u_{e_1} \mathbf{Q}_{e_1})\right)^{v_0 v_0} &= e^{-\Delta t u_{e_1}} = 1 - \frac{\rho - \sigma_1}{\mu^0_{v_0} - \rho + \rho - \sigma_0} \\
&= 1 - \frac{\rho - \sigma_1}{\mu^0_{v_0}} \ , \quad \text{and} \\
\left(\exp(\Delta t u_{e_1} \mathbf{Q}_{e_1})\right)^{v_1 v_0} &= 1 - e^{-\Delta t u_{e_1}} = \frac{\rho - \sigma_1}{\mu^0_{v_0} - \rho + \rho - \sigma_0} \\
&= \frac{\rho - \sigma_1}{\mu^0_{v_0}}.
\end{aligned}$$

From this, it readily follows that

$$\begin{aligned}
\mu_{v_0}(\Delta t) &= \mu^0_{v_0} - \rho + \sigma_1 = \ \mu^0_{v_0} - \rho + \delta_1 \Delta\mu_{v_0} \ , \quad \text{and} \\
\mu_{v_1}(\Delta t) &= \mu^0_{v_1} + \rho - \sigma_1.
\end{aligned}$$

Now suppose that $1 \leq i < s$ is arbitrary but fixed. We consider the following three cases.

Case 1. $S(e_i) = v_i \neq v_0 \neq v_{i+1} = T(e_i)$, and hence $y^0_{v_i} = \mu^0_{v_i}$ and $y^0_{v_{i+1}} = \mu^0_{v_{i+1}}$, and in particular $i \neq s-1$.

The induction hypothesis is that the following equalities hold:

$$\mu_{v_i}(i\Delta t) = \mu^0_{v_i} + \rho - \sigma_i \quad \text{and} \quad \mu_{v_{i+1}}(i\Delta t) = \mu^0_{v_{i+1}}. \tag{35}$$

The second equality is true because $\delta_i = 0$, since we assumed that $i + 1 \neq s$, which implies that there is an edge e_j with $j = i + 1 > i$ such that $S(e_j) = v_j$.

For the $(i+1)^{\text{st}}$ time interval $[i\Delta t, (i+1)\Delta t)$, note that the only nonzero and non-unity entries of the matrix exponential are (given that $y^0_{v_i} = \mu^0_{v_i}$):

$$\begin{aligned}
\left(\exp(\Delta t u_{e_{i+1}} \mathbf{Q}_{e_{i+1}})\right)^{v_i v_i} &= e^{-\Delta t u_{e_{i+1}}} \\
&= 1 - \frac{\rho - \sigma_{i+1}}{\mu^0_{v_i} + \rho - \sigma_i} \ , \quad \text{and} \\
\left(\exp(\Delta t u_{e_{i+1}} \mathbf{Q}_{e_{i+1}})\right)^{v_{i+1} v_i} &= 1 - e^{-\Delta t u_{e_{i+1}}} \\
&= \frac{\rho - \sigma_{i+1}}{\mu^0_{v_i} + \rho - \sigma_i}
\end{aligned}$$

From this, it readily follows that

$$\begin{aligned}
\mu_{v_i}((i+1)\Delta t) &= \mu^0_{v_i}(i\Delta t) - \rho + \sigma_{i+1} = \ \mu^0_{v_i} - \sigma_i + \sigma_{i+1} \\
&= \mu^0_{v_i} + \delta_{i+1} \Delta\mu_{v_i} \ , \quad \text{and} \\
\mu_{v_{i+1}}((i+1)\Delta t) &= \mu^0_{v_{i+1}}(i\Delta t) + \rho - \sigma_{i+1} \\
&= \mu^0_{v_{i+1}} + \rho - \sigma_{i+1} \ ,
\end{aligned}$$

where we have used the recursive definition of σ_j in terms of δ_j.

Case 2. $S(e_i) \neq v_0 = v_{i+1} = T(e_i)$ and $i < s-1$, and hence $y^0_{v_i} = \mu^0_{v_i}$ and $y^0_{v_{i+1}} = \mu^0_{v_{i+1}} - \rho$.

The induction hypothesis is that the following equalities hold:

$$\mu_{v_i}(i\Delta t) = \mu^0_{v_i} + \rho - \sigma_i \quad \text{and} \quad \mu_{v_{i+1}}(i\Delta t) = \mu^0_{v_{i+1}} - \rho.$$

The second equality is true because $\delta_i = 0$, since we assumed that $i < s-1$, which implies that there is an edge e_j with $j = i + 1 > i$ such that $S(e_{j+1}) = v_{j+1}$.

For the $(i+1)^{\text{st}}$ time interval $[i\Delta t, (i+1)\Delta t)$, note that the only nonzero and non-unity entries of the matrix exponential are (given that $y^0_{v_i} = \mu^0_{v_i}$):

$$\begin{aligned}
\left(\exp(\Delta t u_{e_{i+1}} \mathbf{Q}_{e_{i+1}})\right)^{v_i v_i} &= e^{-\Delta t u_{e_{i+1}}} \\
&= 1 - \frac{\rho - \sigma_{i+1}}{\mu^0_{v_i} + \rho - \sigma_i} \ , \quad \text{and} \\
\left(\exp(\Delta t u_{e_{i+1}} \mathbf{Q}_{e_{i+1}})\right)^{v_0 v_i} &= 1 - e^{-\Delta t u_{e_{i+1}}} \\
&= \frac{\rho - \sigma_{i+1}}{\mu^0_{v_i} + \rho - \sigma_i}
\end{aligned}$$

From this, it readily follows that

$$\begin{aligned}
\mu_{v_i}((i+1)\Delta t) &= \mu^0_{v_i}(i\Delta t) - \rho + \sigma_{i+1} \\
&= \mu^0_{v_i} + \sigma_i + \sigma_{i+1} = \mu^0_{v_i} + \delta_{i+1} \Delta\mu_{v_i} \ , \quad \text{and} \\
\mu_{v_0}((i+1)\Delta t) &= \mu^0_{v_0}(i\Delta t) - \rho + \rho - \sigma_{i+1} \\
&= \mu^0_{v_0} + \sigma_{i+1} \ ,
\end{aligned}$$

where we have used the recursive definition of σ_j in terms of δ_j.

Case 3. $S(e_i) = v_i = v_0 \neq T(e_i)$ and $i < s-1$, and hence $y^0_{v_i} = \mu^0_{v_i} - \rho$ and $y^0_{v_{i+1}} = \mu^0_{v_{i+1}}$.

The induction hypothesis is that the following equalities hold:

$$\mu_{v_0}(i\Delta t) = \mu^0_{v_0} - \sigma_i \quad \text{and} \quad \mu_{v_{i+1}}(i\Delta t) = \mu^0_{v_{i+1}}. \tag{36}$$

The second equality is true because $\delta_i = 0$, since we assumed that $i < s-1$, which implies that there is an edge $e_j = e_{i+1}$ such that $S(e_{j+1}) = v_{j+1}$.

For the $(i+1)^{\text{st}}$ time interval $[i\Delta t, (i+1)\Delta t)$, note that the only nonzero and non-unity entries of the matrix exponential are (given that $y^0_{v_i} = \mu^0_{v_i}$):

$$\left(\exp(\Delta t u_{e_{i+1}} \mathbf{Q}_{e_{i+1}})\right)^{v_0 v_0} = e^{-\Delta t u_{e_{i+1}}}$$

$$
\begin{aligned}
&= 1 - \frac{\rho - \sigma_1}{\mu^0_{v_0} - \rho + \rho - \sigma_0} \\
&= 1 - \frac{\rho - \sigma_1}{\mu^0_{v_0}} \ , \quad \text{and} \\
\left(\exp(\Delta t u_{e_{i+1}} \mathbf{Q}_{e_{i+1}})\right)^{v_{i+1}v_0} &= 1 - e^{-\Delta t u_{e_{i+1}}} \\
&= \frac{\rho - \sigma_1}{\mu^0_{v_0} - \rho + \rho - \sigma_0} \\
&= \frac{\rho - \sigma_1}{\mu^0_{v_0}}
\end{aligned}
$$

From this, it readily follows that

$$
\begin{aligned}
\mu_{v_0}((i+1)\Delta t) &= \mu^0_{v_0}(i\Delta t) - \rho + \sigma_{i+1} \\
&= \mu^0_{v_0} - \rho + \sigma_i + \sigma_{i+1} \\
&= \mu^0_{v_0} - \rho + \delta_{i+1}\Delta\mu_{v_{i+1}} \ , \quad \text{and} \\
\mu_{v_{i+1}}((i+1)\Delta t) &= \mu^0_{v_{i+1}}(i\Delta t) + \rho - \sigma_{i+1} \\
&= \mu^0_{v_{i+1}} + \rho - \sigma_{i+1} \ ,
\end{aligned}
$$

where we have used the recursive definition of σ_j in terms of δ_j. By the definition of δ_j, it is clear that after the last edge starting at v_i, $\mu_{v_j}(t)$ stays constant with value $\mu_{v_j}(t) = \mu^0_j + \Delta\mu_i$ for all $(j+1)\Delta t \le t \le T$.

In the final step on the time interval $[T - \Delta t, T]$ corresponding to the edge $e_s = (v_{s-1}, v_s) = (v_{s-1}, v_0)$, the initial conditions are

$$
\begin{aligned}
\mu_{v_{s-1}}((s-1)\Delta t) &= \mu^0_{v_{s-1}} + \rho - \sigma_{s-1} \ , \quad \text{and} \\
\mu_{v_0}((s-1)\Delta t) &= \mu^0_{v_0} - \rho - \delta_s \Delta\mu_{v_0} \\
&= \mu^0_{v_0} - \rho - \Delta\mu_{v_0}.
\end{aligned}
$$

In the second equation, $\delta_s = 1$ because there is no subsequent edge in the path γ that starts from v_0.

For this s^{th} time interval $[T - \Delta t, T]$, note that the only nonzero and non-unity entries of the matrix exponential are (given that $y^0_{v_{s-1}} = \mu^0_{v_{s-1}}$):

$$
\begin{aligned}
\left(\exp(\Delta t u_{e_s} \mathbf{Q}_{e_s})\right)^{v_{s-1}v_{s-1}} &= e^{-\Delta t u_{e_s}} \\
&= 1 - \frac{\rho - \sigma_s}{\mu^0_{v_{s-1}} + \rho - \sigma_{s-1}} \ , \quad \text{and} \\
\left(\exp(\Delta t u_{e_s} \mathbf{Q}_{e_s})\right)^{v_0 v_{s-1}} &= 1 - e^{-\Delta t u_{e_s}} \\
&= \frac{\rho - \sigma_s}{\mu^0_{v_i} + \rho - \sigma_{s-1}}
\end{aligned}
$$

From this, it readily follows that

$$
\begin{aligned}
\mu_{v_{s-1}}(T) &= \mu^0_{v_{s-1}}((s-1)\Delta t) - \rho - \sigma_s \\
&= \mu^0_{v_{s-1}} + \sigma_{s-1} - \sigma_s \\
&= \mu^0_{v_{s-1}} + \Delta\mu_{v_{s-1}} \ , \quad \text{and} \\
\mu_{v_0}(T) &= \mu^0_{v_0}((s-1)\Delta t) - \rho + \rho - \sigma_s \\
&= \mu^0_{v_0} + \sigma_s = \ \mu^0_{v_0} - \sum_{v \in \mathcal{V}\setminus v_0} \Delta\mu_v = \ \mu^0_{v_0} + \Delta\mu_{v_0}.
\end{aligned}
$$

Here, we have used the recursive definition of σ_j in terms of δ_j and the fact that $\sum_{v\in\mathcal{V}} \Delta\mu_v = 0$. □

Theorem IV.17. *Let $T > 0$. If the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is strongly connected, then the system* (7) *is globally controllable within T from every point in the interior of the simplex $\mathcal{P}(\mathcal{V})$.*

Proof. Suppose that $\boldsymbol{\mu}^0, \boldsymbol{\mu}^T \in \text{int}(\mathcal{P}(\mathcal{V}))$. Let $\rho = \frac{1}{2}\min\{\mu^0_v, \mu^T_v : v \in \mathcal{V}\}$, $L = \|\boldsymbol{\mu}^T - \boldsymbol{\mu}^0\|_1$, and $N = \text{ceil}(L/\rho)$. Partition the straight-line segment from $\boldsymbol{\mu}^0$ to $\boldsymbol{\mu}^T$ into N segments, e.g. with endpoints $\mathbf{y}^k = \boldsymbol{\mu}^0 + \frac{k}{N}(\boldsymbol{\mu}^T - \boldsymbol{\mu}^0)$ for $0 \le k \le N$. Using Proposition IV.16, there exist controls $\mathbf{u}^k : [\frac{kT}{N}, \frac{(k+1)T}{N}] \mapsto [0,\infty)^{N_\mathcal{E}}$ that successively steer the system from $\boldsymbol{\mu}^k$ to $\boldsymbol{\mu}^{k+1}$. Thus, the concatenation of these controls steers the system from $\boldsymbol{\mu}^0$ to $\boldsymbol{\mu}^T$ in time T using piecewise constant controls that take values only on the axes of $\mathbb{R}^M_+$. □

C. Controllability of the PDE System (11)

In this section, we will consider the controllability problem described in Problem III.3. We define some new notation that will be needed in this section and the following one. These definitions will be used to construct solutions of the system of PDEs (9) and hence enable the controllability and stability analysis. Let $\mathbf{a} = [a_1\ a_2\ ...\ a_N]^T \in \mathbf{L}^\infty(\Omega) = Z_1 \times ... \times Z_N$, where $a_i \in L^\infty(\Omega)$ and $Z_i = L^\infty(\Omega)$ for each $i \in \mathcal{V}$. If $c > 0$, then we write $\mathbf{a} \ge c$ to denote that $a_i \ge c$ for each $i \in \mathcal{V}$. We will assume throughout that this condition is satisfied by $\mathbf{a}$ for some positive constant c. We consider the operator $\boldsymbol{\mathcal{B}}_\mathbf{a} : \mathcal{D}(\boldsymbol{\mathcal{B}}_\mathbf{a}) \to \mathbf{L}^2_\mathbf{a}(\Omega)$, where $\mathbf{L}^2_\mathbf{a}(\Omega) = L^2_{a_1}(\Omega) \times ... \times L^2_{a_N}(\Omega)$ is equipped with the norm $\|\cdot\|_\mathbf{a}$, defined as $\|\mathbf{u}\|_\mathbf{a} = (\sum_{i=1}^N \|u_i\|^2_a)^{1/2}$ for each $\mathbf{u} = [u_1\ ...\ u_N]^T \in \mathbf{L}^2_\mathbf{a}(\Omega)$, and $\mathcal{D}(\boldsymbol{\mathcal{B}}_\mathbf{a}) = \mathcal{D}(B_{a_1}) \times \mathcal{D}(B_{a_2}) \times ... \times \mathcal{D}(B_{a_N})$. The operator $\boldsymbol{\mathcal{B}}_\mathbf{a}$ is defined by $\boldsymbol{\mathcal{B}}_\mathbf{a}\mathbf{v} = [B_{a_1}v_1\ B_{a_2}v_2\ ...\ B_{a_N}v_N]^T$ for each $\mathbf{v} = [v_1\ ...\ v_N]^T \in \mathcal{D}(\boldsymbol{\mathcal{B}}_\mathbf{a})$. Corresponding to each matrix $\mathbf{Q}_e$, we associate a bounded operator $\boldsymbol{\mathcal{Q}}_e$ on the space $\mathbf{L}^2_\mathbf{a}(\Omega)$ given by $(\boldsymbol{\mathcal{Q}}_e\mathbf{y})(\mathbf{x}) = \mathbf{Q}_e\mathbf{y}(\mathbf{x})$ for each $\mathbf{y} = [y_1\ ...\ y_N]^T \in \mathbf{L}^2_\mathbf{a}(\Omega)$ and a.e. $\mathbf{x} \in \Omega$. Let $b \in \mathbf{L}^\infty(\Omega)$. $\boldsymbol{\mathcal{M}}_\mathbf{b}$ will denote the multiplication operator defined by $\boldsymbol{\mathcal{M}}_\mathbf{b}\mathbf{v} = [\mathcal{M}_{b_1}v_1\ \mathcal{M}_{b_2}v_2\ ...\ \mathcal{M}_{b_N}v_N]^T$ for each $\mathbf{v} \in \mathbf{L}^2(\Omega) = \mathbf{L}^2_\mathbf{a}(\Omega)$. For a function $K_e \in L^\infty(\Omega)$, $K_e\boldsymbol{\mathcal{Q}}_e$ will denote the product operator $\boldsymbol{\mathcal{M}}_\mathbf{b}\boldsymbol{\mathcal{Q}}_e$, where $\boldsymbol{\mathcal{M}}_\mathbf{b}$ is the multiplication operator corresponding to the function $\mathbf{b} \in \mathbf{L}^\infty(\Omega)$ defined by setting $b_i = K_e$ for each $i \in \mathcal{V}$.

Using these definitions, we construct some semigroups on the space $\mathbf{L}^2_\mathbf{a}(\Omega)$ that will be used to address Problem III.3.

Lemma IV.18. *Let $\{K_e\}_{e\in\mathcal{E}}$ be a set of non-negative functions in $L^\infty(\Omega)$. Suppose $\mathbf{b} \in \mathbf{L}^\infty(\Omega)$ such that $b_i = D_i\mathbf{1}$ is a positive constant function for each $i \in \mathcal{V}$. Then the operator $-\boldsymbol{\mathcal{M}}_\mathbf{b}\boldsymbol{\mathcal{B}}_\mathbf{a} + \sum_{e\in\mathcal{E}} K_e\boldsymbol{\mathcal{Q}}_e$ generates a semigroup of operators $(\boldsymbol{\mathcal{S}}(t))_{t\ge0}$ on $\mathbf{L}^2_\mathbf{a}(\Omega)$. Moreover, the semigroup is positive and mass-conserving, i.e., if $\mathbf{y}^0 \in \mathbf{L}^2_\mathbf{a}(\Omega)$ is real-valued, then $\sum_{i\in\mathcal{V}}\int_\Omega(\boldsymbol{\mathcal{S}}(t)\mathbf{y}^0)_i(\mathbf{x})d\mathbf{x} = \sum_{i\in\mathcal{V}}\int_\Omega y_i(\mathbf{x})d\mathbf{x}$ for all $t \ge 0$.*

Additionally, if $a_i \in W^{1,\infty}(\Omega)$, then $\boldsymbol{\mathcal{S}}(t)\mathbf{y}^0$ is the unique mild solution of the system (9) *with $f_i = 1/a_i$, $\mathbf{v}_i(\cdot,t) = D_i\nabla f_i/f_i$, and $u_e(t) = K_e$ for all $i \in \mathcal{V}$, all $e \in \mathcal{E}$, and all $t \in [0, t_f]$.*

Proof. The generation of the semigroup $(\boldsymbol{\mathcal{S}}(t))_{t\ge0}$ follows from the fact that $-\boldsymbol{\mathcal{M}}_\mathbf{b}\boldsymbol{\mathcal{B}}_\mathbf{a} + \sum_{e\in\mathcal{E}} K_e\boldsymbol{\mathcal{Q}}_e$ is a bounded perturbation of the operator $-\boldsymbol{\mathcal{M}}_\mathbf{b}\boldsymbol{\mathcal{B}}_\mathbf{a}$. The positivity preserving property of the semigroup can be demonstrated as follows using the *Lie-Trotter product formula* [17][Corollary III.5.8]. Let $(\boldsymbol{\mathcal{U}}(t))_{t\ge0}$ be the semigroup generated by the

operator $\sum_{e\in\mathcal{E}} K_e\boldsymbol{\mathcal{Q}}_e$. In fact, the semigroup can be explicitly represented as $\boldsymbol{\mathcal{U}}(t) = e^{\sum_{e\in\mathcal{E}} K_e\boldsymbol{\mathcal{Q}}_e t}$ for each $t \geq 0$. Moreover, $(e^{\sum_{e\in\mathcal{E}} K_e\boldsymbol{\mathcal{Q}}_e t}\mathbf{y}^0)(\mathbf{x}) = e^{\sum_{e\in\mathcal{E}} K_e(\mathbf{x})\mathbf{Q}_e t}\mathbf{y}^0(\mathbf{x})$ for each $\mathbf{y}^0 \in \mathbf{L}^2_{\mathbf{a}}(\Omega)$ and a.e. $\mathbf{x} \in \Omega$. The semigroup $(\boldsymbol{\mathcal{U}}(t))_{t\geq 0}$ is positivity preserving since each matrix $\mathbf{Q}_e$ has positive off-diagonal entries. From Corollary IV.2, we also note that the semigroup $(\boldsymbol{\mathcal{V}}(t))_{t\geq 0}$ generated by the operator $-\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}}$ is positivity preserving. Moreover, since $\sum_{e\in\mathcal{E}} K_e\boldsymbol{\mathcal{Q}}_{\boldsymbol{e}}$ is a bounded operator, there exists $w \in \mathbb{R}$ such that $\|\boldsymbol{\mathcal{U}}(t)\|_{op} \leq Me^{wt}$ for some positive constant M for all $t \geq 0$. The semigroup $(\boldsymbol{\mathcal{V}}(t))_{t\geq 0}$ is *contractive*, i.e., $\|\boldsymbol{\mathcal{V}}(t)\|_{op} \leq 1$ for all $t \geq 0$. Hence, it follows from the Lie-Trotter product formula that $\boldsymbol{\mathcal{S}}(t)\mathbf{y}^0 = \lim_{n\to\infty}[\boldsymbol{\mathcal{V}}(t/n)\boldsymbol{\mathcal{U}}(t/n)]^n\mathbf{y}^0$ for all $t \geq 0$. Therefore, the semigroup $(\boldsymbol{\mathcal{S}}(t))_{t\geq 0}$ is positivity preserving. Through another application of the Lie-Trotter product formula, it follows that $\sum_{i\in\mathcal{V}}\int_\Omega(\boldsymbol{\mathcal{S}}(t)\mathbf{y}^0)_i(\mathbf{x})d\mathbf{x} = \sum_{i\in\mathcal{V}}\int_\Omega y_i^0(\mathbf{x})d\mathbf{x}$ for all $t \geq 0$. □

In the following lemma, we identify a relation between solutions of the system of PDEs (9) and solutions of the ODE (7).

Lemma IV.19. *Let $\{q_e\}_{e\in\mathcal{E}}$ be a set of non-negative constants. Suppose $\mathbf{b} \in \mathbf{L}^\infty(\Omega)$ such that $b_i = D_i\mathbf{1}$ is a positive constant function for each $i \in \mathcal{V}$. Let $(\boldsymbol{\mathcal{S}}(t))_{t\geq 0}$ be the semigroup generated by the operator $-\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}} + \sum_{e\in\mathcal{E}} q_e\boldsymbol{\mathcal{Q}}_e$. Additionally, assume that $\mathbf{y}^0 \in \mathcal{D}(-\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}})$ is real-valued and that $\mu_i^0 = \int_\Omega y_i^0(\mathbf{x})d\mathbf{x}$ for each $i \in \mathcal{V}$. Then the solution of the system (7) satisfies $\mu_i(t) = \int_\Omega(\boldsymbol{\mathcal{S}}(t)\mathbf{y}^0)_i(\mathbf{x})d\mathbf{x}$ for each $t \geq 0$ and each $i \in \mathcal{V}$.*

Proof. Let $\mathbf{y}(\cdot,t) = \boldsymbol{\mathcal{S}}(t)\mathbf{y}^0$ for each $t \geq 0$. Then the result follows by noting that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega y_i(\mathbf{x},t)d\mathbf{x}\\
&\quad= \int_\Omega D_iB_{a_i}y_i(\mathbf{x},t)d\mathbf{x} + \sum_{j=1}^N\sum_{e\in\mathcal{E}}\int_\Omega q_eQ_e^{ij}y_i(\mathbf{x},t)d\mathbf{x}\\
&\quad= D_i\sigma_{a_i}(y_i,1/a_i) + \sum_{j=1}^N\sum_{e\in\mathcal{E}} q_eQ_e^{ij}\int_\Omega y_i(\mathbf{x},t)d\mathbf{x}\\
&\quad= \sum_{j=1}^N\sum_{e\in\mathcal{E}} q_eQ_e^{ij}\int_\Omega y_i(\mathbf{x},t)d\mathbf{x}
\end{aligned}$$

for all $t \geq 0$. □

The lemma above allows us to apply the results of Theorems IV.13 and IV.17 to prove the following controllability result, which addresses Problem III.3.

Theorem IV.20. *Let Ω be a domain that is $C^{1,1}$ or convex and that satisfies the chain condition. Let $t_f > 0$. Let $f_i \in W^{1,\infty}(\Omega)$ for each $i \in \mathcal{V}$ such that $f_i \geq c$ for some positive constant c. Suppose $\mathbf{y}^0 \in \mathbf{L}^2(\Omega)$ such that $\mathbf{y}^0 \geq 0$ and $\sum_i\int_\Omega f_i(\mathbf{x})d\mathbf{x} = \sum_i\int_\Omega y_i^0(\mathbf{x})d\mathbf{x}$. Then there exist control parameters $\{v_i\}_{i\in V}$ in $L^\infty([0,t_f];L^\infty(\Omega)^n)$ and $\{u_e\}_{e\in\mathcal{E}}$ in $L^\infty([0,t_f])$, where each u_e is non-negative, such that the unique mild solution of the system (9) satisfies $y_i(\cdot,t_f) = f_i$ for each $i \in \mathcal{V}$.*

Proof. Let $\mathbf{v}_i(\cdot,t) = \mathbf{0}$ for each $t \in [0,t_f/2]$ and for each $i \in \mathcal{V}$. Then from Theorem IV.17 and Lemmas IV.18 and IV.19, it follows that there exist piecewise constant parameters $u_e : [0,t_f/2] \to \mathbb{R}_+$ such that the mild solution of the PDE (9) satisfies $\int_\Omega y_i(\mathbf{x},t_f/2)d\mathbf{x} = \int_\Omega f_i(\mathbf{x})d\mathbf{x}$ for each $i \in \mathcal{V}$. Then the result follows by extending the function u_e to the domain $[0,t_f]$ by defining $u_e(t) = 0$ for $t \in (t_f/2,t_f]$ and by defining $\mathbf{v}_i(\cdot,t)$ for $t \in (t_f/2,t_f]$ as in the proof of Theorem IV.13. □

V. Stabilizing Desired Stationary Distributions

In this section, we address Problem III.7. We first briefly review the notion of *irreducibility* of a *positive operator* [31], which will be used extensively in the theorems in this section. Let $\mathcal{P}$ be a positive operator on the Hilbert space $X = L^2_a(\Omega)$ (or $\mathbf{L}^2_{\mathbf{a}}(\Omega)$) for some $a \in L^\infty(\Omega)$ (or $\mathbf{a} \in \mathbf{L}^\infty(\Omega)$), i.e., a linear bounded operator that maps real-valued non-negative elements of X to real-valued non-negative elements of X. Let $\tilde{\Omega} \subset \Omega$ (or $\tilde{\Omega} \subset \Omega^N$) be a measurable subset. Consider the set $\mathcal{I}_{\tilde{\Omega}}$ defined by $\mathcal{I}_{\tilde{\Omega}} = \big\{f \in X : \tilde{\Omega} \subset \{\mathbf{x} \in \Omega : f(\mathbf{x}) = 0\}\big\}$. $\mathcal{P}$ will be called *irreducible* if the only measurable sets $\tilde{\Omega} \subset \Omega$ for which the set $\mathcal{I}_{\tilde{\Omega}}$ is invariant under $\mathcal{P}$ are $\tilde{\Omega} = \Omega$ (or Ω^N) and $\tilde{\Omega} = \varnothing$, the null set. A positive semigroup of operators $(\mathcal{T}(t))_{t\geq 0}$ on X will be called irreducible if $\mathcal{T}(t)$ is an irreducible operator for every $t > 0$. Suppose that A is the generator of the semigroup $(\mathcal{T}(t))_{t\geq 0}$ and $s(A) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{spec}(A)\}$. Then $(\mathcal{T}(t))_{t\geq 0}$ being irreducible is equivalent to $(\lambda\mathbb{I} - A)^{-1}$ mapping real-valued non-negative elements of X to strictly positive elements of X for every $\lambda > s(A)$ [2][Definition C-III.3.1]. Note that the definitions in the cited reference are stated in a general framework of *Banach lattices*, for which $(\mathcal{T}(t))_{t\geq 0}$ being irreducible is equivalent to $(\lambda\mathbb{I} - A)^{-1}$ mapping positive elements of X to *quasi-interior* elements of X. However, for the spaces that we consider, quasi-interior elements are the same as functions that are positive almost everywhere on their domain of definition.

Theorem V.1. *Let $\{q_e\}_{e\in\mathcal{E}}$ be a set of non-negative constants. Then $\mathrm{spec}(\sum_{e\in\mathcal{E}} q_e\mathbf{Q}_e) \subset \bar{\mathbb{C}}_-$.*

Proof. This follows from [33][Theorem II.1.1] by noting that all the elements of the matrix $\mathbf{G}_\lambda = \lambda\mathbb{I} + \sum_{e\in\mathcal{E}} q_e\mathbf{Q}_e$ are non-negative for $\lambda > 0$ large enough. □

Theorem V.2. *Let $\{q_e\}_{e\in\mathcal{E}}$ be a set of non-negative constants. Let $\mathbf{a} \in \mathbf{L}^\infty(\Omega)$ such that $\mathbf{a} \geq c$ for some positive constant c. Suppose $\mathbf{b} \in \mathbf{L}^\infty(\Omega)$ such that $b_i = D_i\mathbf{1}$ is a positive constant function for each $i \in \mathcal{V}$. Then $\mathrm{spec}(-\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}} + \sum_{e\in\mathcal{E}} q_ea_{S(e)}\boldsymbol{\mathcal{Q}}_e) \subset \bar{\mathbb{C}}_-$.*

Proof. Let $\boldsymbol{\mathcal{W}} = -\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}} + \sum_{e\in\mathcal{E}} q_ea_{S(e)}\boldsymbol{\mathcal{Q}}_e$. Let $\lambda \in \mathbb{C}\backslash\mathrm{spec}(-\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}} + \sum_{e\in\mathcal{E}} q_ea_{S(e)}\boldsymbol{\mathcal{Q}}_e)$ be real and large enough such that $(\lambda\mathbb{I} - \boldsymbol{\mathcal{W}})^{-1}$ is a positive operator, i.e., $(\lambda\mathbb{I} - \boldsymbol{\mathcal{W}})^{-1}f \geq 0$ whenever $f \geq 0$. Such a λ necessarily exists because the semigroup $(\boldsymbol{\mathcal{U}}(t))_{t\geq 0}$ generated by the operator $\boldsymbol{\mathcal{W}}$ is positivity preserving from Lemma IV.18. Hence, the existence of λ follows from the resolvent formula $(\lambda\mathbb{I} - \boldsymbol{\mathcal{W}})^{-1} = \int_0^\infty e^{-\lambda t}\boldsymbol{\mathcal{U}}(t)dt$ when λ is greater than the growth

bound of the semigroup $(\boldsymbol{\mathcal{U}}(t))_{t\geq 0}$, which is equal to the spectral growth bound $s(\boldsymbol{\mathcal{W}})$ of $\boldsymbol{\mathcal{W}}$ since $(\boldsymbol{\mathcal{U}}(t))_{t\geq 0}$ is analytic [17][Theorem II.1.10]. Let $R_\lambda = (\lambda\mathbb{I} - \boldsymbol{\mathcal{W}})^{-1}$. The operator $-\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}}$ has a compact resolvent since $H^1_{a_i}(\Omega)$ is compactly embedded in $L^2_{a_i}(\Omega)$ for each $i \in \mathcal{V}$ [38][Proposition 10.6]. The operator R_λ is compact and positivity preserving since $\boldsymbol{\mathcal{W}}$ is a bounded perturbation of $-\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}}$. Additionally, the spectral radius of R_λ is positive since 0 is an eigenvalue of $\boldsymbol{\mathcal{W}}$ (and hence $\frac{1}{\lambda}$ is an eigenvalue of R_λ). Therefore, from the *Krein-Rutman theorem* [31][Theorem 4.1.4], it follows that if r is the spectral radius of the operator R_λ, then there exists a positive nonzero element $\mathbf{h} \in \mathbf{L}^2_{\mathbf{a}}(\Omega)$ such that $r\mathbf{h} - R_\lambda\mathbf{h} = \mathbf{0}$. Then it follows that $\mathbf{h} \in \mathcal{D}(\boldsymbol{\mathcal{W}})$ and $(\lambda - \frac{1}{r})\mathbf{h} - \boldsymbol{\mathcal{W}}\mathbf{h} = \mathbf{0}$. For the sake of contradiction, suppose that $\lambda > \frac{1}{r}$. Then we have that

$$\alpha \int_\Omega h_i(\mathbf{x})d\mathbf{x} \; + \; \int_\Omega D_i(B_{a_i}h_i)(\mathbf{x})d\mathbf{x} \\ - \; \sum_{e\in\mathcal{E}}\sum_{j=1}^N \int_\Omega q_e a_{S(e)}(\mathbf{x})Q_e^{ij}h_j(\mathbf{x})d\mathbf{x} = 0$$

for each $i \in \mathcal{V}$, where $\alpha = \lambda - \frac{1}{r}$. This implies that

$$\alpha \int_\Omega h_i(\mathbf{x})d\mathbf{x} - \sum_{e\in\mathcal{E}}\sum_{j=1}^N \int_\Omega q_e a_{S(e)}(\mathbf{x})Q_e^{ij}h_j(\mathbf{x})d\mathbf{x} = 0$$

for each $i \in \mathcal{V}$. But this implies that the matrix $\sum_{e\in\mathcal{E}} q_e k_{S(e)}\mathbf{Q}_e$, where the constants $\{k_i\}_{i\in\mathcal{V}}$ are such that

$$\int_\Omega a_i(\mathbf{x})h_i(\mathbf{x})d\mathbf{x} = k_i \int_\Omega h_i(\mathbf{x})d\mathbf{x},$$

has a positive eigenvalue α. This contradicts Theorem V.1, since $\mathrm{spec}(\sum_{e\in\mathcal{E}} q_e k_{S(e)}\mathbf{Q}_e) \subset \bar{\mathbb{C}}_-$. □

Proposition V.3. *Let $\mathcal{G}$ be strongly connected. Let $\mathbf{f} \in \mathbf{L}^\infty(\Omega)$ be such that $\mathbf{f} \geq c$ for some positive constant c. Let $\mathbf{b} \in \mathbf{L}^\infty(\Omega)$ such that $b_i = D_i\mathbf{1}$ is a positive constant function for each $i \in \mathcal{V}$. Suppose $\mathbf{y}^0 \in \mathbf{L}^2(\Omega)$ such that $\mathbf{y}^0 \geq 0$ and $\sum_i \int_\Omega f_i(\mathbf{x})d\mathbf{x} = \sum_i \int_\Omega y_i^0(\mathbf{x})d\mathbf{x} = 1$. Let $\mathbf{a} \in \mathbf{L}^\infty(\Omega)$ be such that $a_i = 1/f_i$ for each $i \in \mathcal{V}$. Then there exist positive parameters $\{q_e\}_{e\in\mathcal{E}}$ such that, if $(\boldsymbol{\mathcal{V}}(t))_{t\geq 0}$ is the semigroup generated by the operator $-\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}} + \sum_{e\in\mathcal{E}} q_e a_{S(e)}\boldsymbol{\mathcal{Q}}_e$, then we have*

$$\|\boldsymbol{\mathcal{V}}(t)\mathbf{y}^0 - \mathbf{f}\|_2 \; \leq \; Me^{-\lambda t} \tag{37}$$

for some positive constants M and λ and all $t \geq 0$.

Proof. Since the graph $\mathcal{G}$ is assumed to be strongly connected, from [14][Theorem IV.5] (see [7] for proof) we know that there exist positive parameters $\{q_e\}_{e\in\mathcal{E}}$ such that, if $u_e(t) = q_e$ for all $e \in \mathcal{E}$ and all $t \geq 0$, then the solution $\boldsymbol{\mu}(t)$ of the system (7) satisfies

$$\|\boldsymbol{\mu}(t) - \boldsymbol{\mu}^{eq}\|_2 \; \leq \; M_1 e^{-\lambda_1 t} \tag{38}$$

for some positive constants M_1 and λ_1 and all $t \geq 0$, where $\mu_k^{eq} = \int_\Omega f_k(\mathbf{x})d\mathbf{x}$ for each $k \in \mathcal{V}$ and $\boldsymbol{\mu}^0 \in \mathcal{P}(\mathcal{V})$. In particular, 0 is a simple eigenvalue of the irreducible operator $\sum_{e\in\mathcal{E}} q_e\mathbf{Q}_e$ and $\boldsymbol{\mu}^{eq}$ is the corresponding unique (up to a scalar multiple) and strictly positive eigenvector. Then 0 is an eigenvalue for the operator $\boldsymbol{\mathcal{W}} = -\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}} + \sum_e q_e a_{S(e)}\boldsymbol{\mathcal{Q}}_e$ with the corresponding eigenvector $\mathbf{f}$, by construction. We will show that this eigenvalue is simple and is the dominant eigenvalue. Let $\mathbf{g} \in \mathbf{L}^2_{\mathbf{a}}(\Omega)$ such that $\mathbf{g}$ is not the zero element $\mathbf{0}$ and is non-negative a.e. in Ω^N. Defining $\mathbf{h} = (\lambda\mathbb{I} - \boldsymbol{\mathcal{W}})^{-1}\mathbf{g}$ for some $\lambda > 0$, we have that

$$\lambda \int_\Omega h_i(\mathbf{x})d\mathbf{x} + \int_\Omega D_i(B_{a_i}h_i)(\mathbf{x})d\mathbf{x} \\ - \sum_{e\in\mathcal{E}}\sum_{j=1}^N \int_\Omega q_e a_{S(e)}(\mathbf{x})Q_e^{ij}h_j(\mathbf{x})d\mathbf{x} = \int_\Omega g_i(\mathbf{x})d\mathbf{x}$$

for each $i \in \mathcal{V}$. This implies that

$$\lambda \int_\Omega h_i(\mathbf{x})d\mathbf{x} \; - \; \sum_{e\in\mathcal{E}}\sum_{j=1}^N \int_\Omega q_e a_{S(e)}(\mathbf{x})Q_e^{ij}h_j(\mathbf{x})d\mathbf{x} \\ = \int_\Omega g_i(\mathbf{x})d\mathbf{x}$$

for each $i \in \mathcal{V}$, which implies that

$$\lambda \int_\Omega h_i(\mathbf{x})d\mathbf{x} - \sum_{e\in\mathcal{E}}\sum_{j=1}^N \int_\Omega q_e k_{S(e)}Q_e^{ij}h_j(\mathbf{x})d\mathbf{x} = \int_\Omega g_i(\mathbf{x})d\mathbf{x} \tag{39}$$

for each $i \in \mathcal{V}$ for some positive constants $k_i > 0$. The existence of such positive constants is guaranteed, since we assumed that $\mathbf{f}$ is non-negative and hence $\mathbf{h}$ is non-negative. However, $\sum_e q_e k_{S(e)}\mathbf{Q}_e$ generates an irreducible semigroup on $\mathbb{R}^N$ whenever $q_e > 0$ implies that $k_e > 0$ for all $e \in \mathcal{E}$. Hence, $(\lambda\mathbb{I} - \sum_e q_e k_{S(e)}\mathbf{Q}_e)^{-1}$ maps non-negative, nonzero elements of $\mathbb{R}^N$ to strictly positive elements of $\mathbb{R}^N$. This implies that $\int_\Omega h_i(\mathbf{x})d\mathbf{x} > 0$ for each $i \in \mathcal{V}$. From this, we can conclude that $h_i(\mathbf{x}) > 0$ for a.e. $\mathbf{x} \in \Omega$ for each $i \in \mathcal{V}$. To see this more explicitly, note that $\mathbf{h}$ must satisfy

$$\lambda h_i - D_i B_{a_i} h_i - G^{ii}a_i h_i = g_i + \sum_{j=1, j\neq i}^N G^{ij}a_j h_j \tag{40}$$

for each $i \in \mathcal{V}$, where $\mathbf{G} = \sum_{e\in\mathcal{E}} q_e\mathbf{Q}_e$. Let $\mathcal{M}_{a_i}$ be the multiplication operator, defined on $L^2(\Omega) = L^2_{a_i}(\Omega)$, that is associated with the function a_i. Since $a_i \geq \ell$ for some $\ell > 0$, the inverse $R^i_\lambda = (\lambda\mathbb{I} - D_i B_{a_i} - G^{ii}\mathcal{M}_{a_i})^{-1} = (\lambda\mathcal{M}^{-1}_{a_i} - D_i B_{a_i}\mathcal{M}^{-1}_{a_i} - G^{ii}\mathbb{I})^{-1}\mathcal{M}^{-1}_{a_i}$ exists. The operator $\lambda\mathcal{M}^{-1}_{a_i} - D_i B_{a_i}\mathcal{M}^{-1}_{a_i}$ generates an irreducible semigroup on $L^2(\Omega)$ [35][Theorem 4.5] (see equation (4.8) in the cited reference for the class of operators considered); formally, $B_{a_i}\mathcal{M}^{-1}_{a_i}$ is the operator $\nabla \cdot (f_i\nabla(\cdot))$. Hence, $(R^i_\lambda[g_i + \sum_{j=1, j\neq i}^N G^{ij}a_j h_j])(\mathbf{x})$ is strictly positive for a.e. $\mathbf{x} \in \Omega$ and each $i \in \mathcal{V}$, since $\sum_{j=1, j\neq i}^N G^{ij}$ and h_i are nonzero for each $i \in \mathcal{V}$. Therefore, $(\lambda\mathbb{I} - \boldsymbol{\mathcal{W}})^{-1}$ maps nonzero, non-negative elements of $\mathbf{L}^2_{\mathbf{a}}(\Omega)$ to strictly positive elements of $\mathbf{L}^2_{\mathbf{a}}(\Omega)$. This implies that the semigroup generated by the operator $\boldsymbol{\mathcal{W}}$ is irreducible. Now, we can use [2][Corollary C-III.3.17] to establish that the eigenvalue 0 is simple and is the dominant eigenvalue. This follows from the cited corollary because $\boldsymbol{\mathcal{W}}$ has a compact resolvent and generates an analytic semigroup, due to the fact that it is a bounded perturbation of the operator $-\boldsymbol{\mathcal{M}}_{\mathbf{b}}\boldsymbol{\mathcal{B}}_{\mathbf{a}}$, which itself has a compact resolvent and generates an analytic semigroup [17][Proposition III.1.12]. Additionally,

we know from [17][Corollary III.1.19] that since $\boldsymbol{\mathcal{W}}$ has a compact resolvent, its spectrum is discrete. Then the result follows from [17][Corollary V.3.3]. □

Irreducibility is not necessary, but only sufficient, for the simplicity of the dominant eigenvalue of a compact positive operator. The goal of the following proposition and theorem is to extend the result in Proposition V.3 to a much larger set of equilibrium distributions, for which the resulting semigroup is not necessarily irreducible.

Proposition V.4. *Let $\mathbf{P} \in \mathbb{R}^{N\times N}$ be essentially non-negative, i.e., $P^{ij} \geq 0$ for all $i \neq j$ in $\mathcal{V}$. Let $\boldsymbol{\mathcal{P}}$ be the linear bounded operator on $\mathbf{L}^2(\Omega)$, defined pointwise using $\mathbf{P}$ as $(\boldsymbol{\mathcal{P}}\mathbf{h})(\mathbf{x}) = \mathbf{P}\mathbf{h}(\mathbf{x})$ for a.e. $\mathbf{x} \in \Omega$ for all $\mathbf{h} \in \mathbf{L}^2(\Omega)$. Suppose $\mathbf{b} \in \mathbf{L}^\infty(\Omega)$ such that $b_i = D_i\mathbf{1}$ is a positive constant function for each $i \in \mathcal{V}$. In addition, suppose that $\mathrm{spec}(\mathbf{P})$ lies in $\mathbb{C}_-$. If $a_i = \mathbf{1}$ for each $i \in \mathcal{V}$, then $\mathrm{spec}(-\boldsymbol{\mathcal{M}}_\mathbf{b}\boldsymbol{\mathcal{B}}_\mathbf{a} + \boldsymbol{\mathcal{P}})$ lies in $\mathbb{C}_-$.*

Proof. The proof follows the same line of argument as Theorem V.2. Note that according to the Lie-Trotter product formula, $\boldsymbol{\mathcal{W}} = -\boldsymbol{\mathcal{M}}_\mathbf{b}\boldsymbol{\mathcal{B}}_\mathbf{a} + \boldsymbol{\mathcal{P}}$ generates a positive semigroup since both $-\boldsymbol{\mathcal{B}}_\mathbf{a}$ and $\boldsymbol{\mathcal{P}}$ generate positivity preserving semigroups. Hence, if $\lambda > 0$ is large enough, then $R_\lambda = (\lambda - \boldsymbol{\mathcal{W}})^{-1}$ is a positive operator. Moreover, R_λ is a compact operator and has a nonzero spectral radius r. From the Krein-Rutman theorem [31][Theorem 4.1.4], it follows that there exists a positive function $\mathbf{h} \in \mathbf{L}^2_\mathbf{a}(\Omega) = \mathbf{L}^2(\Omega)$ such that $r\mathbf{h} - R_\lambda\mathbf{h} = \mathbf{0}$. This implies that $\lambda - \frac{1}{r}$ is an eigenvalue of $\boldsymbol{\mathcal{W}}$. However, this implies that $\int_\Omega -(B_{a_i}h_i) = 0$ for each $i \in \mathcal{V}$, and hence that

$$\left(\lambda - \frac{1}{r}\right)\int_\Omega h_i(\mathbf{x})d\mathbf{x} - \sum_{j=1}^N \int_\Omega P^{ij}h_j(\mathbf{x})d\mathbf{x} \;=\; 0$$

for each $i \in \mathcal{V}$. If $\lambda - \frac{1}{r} \geq 0$, then we arrive at a contradiction, since $\mathrm{spec}(\mathbf{P})$ lies in $\mathbb{C}_-$. Here, we have used the fact that $\mathbf{h}$ is a positive function, and therefore $\int_\Omega h_i(\mathbf{x})d\mathbf{x}$ cannot be equal to 0 for each $i \in \mathcal{V}$. □

Theorem V.5. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be strongly connected, and let Ω be an extension domain. Let $\mathbf{b} \in \mathbf{L}^\infty(\Omega)$ such that $b_i = D_i\mathbf{1}$ is a positive constant function for each $i \in \mathcal{V}$. Let $\mathbf{f} \in \mathbf{L}^\infty(\Omega)$ be such that $f_i \geq c\int_\Omega f_i(\mathbf{x})d\mathbf{x}$ for some positive constant $c > 0$. Let $\mathcal{V}_1 = \{i \in \mathcal{V} : \int_\Omega f_i(\mathbf{x})d\mathbf{x} > 0\}$. Additionally, consider the set $\mathcal{E}_1 = \{e \in \mathcal{E} \;:\; S(e), T(e) \in \mathcal{V}_1\}$. Suppose that the graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ is strongly connected. Then there exist $\mathbf{a} \in \mathbf{L}^\infty(\Omega)$ and spatially-dependent reaction coefficients $\{K_e(\mathbf{x})\}_{e\in\mathcal{E}} \in \mathbf{L}^\infty(\Omega)$ for which $-\boldsymbol{\mathcal{M}}_\mathbf{b}\boldsymbol{\mathcal{B}}_\mathbf{a} + \sum_{e\in\mathcal{E}} K_e\boldsymbol{\mathcal{Q}}_e$ generates a positive semigroup $(\boldsymbol{\mathcal{S}}(t))_{t\geq 0}$ on $\mathbf{L}^2_\mathbf{a}(\Omega)$ such that if $\mathbf{y}^0 \in \mathbf{L}^2_\mathbf{a}(\Omega)$ is a positive function and $\sum_{i\in\mathcal{V}}\int_\Omega f_i(\mathbf{x})d\mathbf{x} = \sum_{i\in\mathcal{V}}\int_\Omega y_i(\mathbf{x})d\mathbf{x}$, then*

$$\|\boldsymbol{\mathcal{S}}(t)\mathbf{y}_0 - \mathbf{f}\| \;\leq\; Me^{-\lambda t} \tag{41}$$

for some positive constants M and λ and all $t \geq 0$.

Proof. Without loss of generality, we assume that the set $\mathcal{V}_1$ is of the form $\mathcal{V}_1 = \{1, 2, ..., \bar{N}\}$ for some integer $\bar{N} \leq N$. Define $\boldsymbol{\mu}^{eq} \in \mathbb{R}^N_+$ such that $\mu^{eq}_i = \int_\Omega f_i(\mathbf{x})d\mathbf{x}$ for each $i \in \mathcal{V}$. Then from [14][Theorem IV.5] (see [7] for proof), it follows that there exist positive constants $\{q_e\}_{e\in\mathcal{E}}$ such that the solution $\boldsymbol{\mu}(t)$ of the ODE system (7) converges exponentially to $\boldsymbol{\mu}^{eq}$. In particular, the matrix $\sum_{e\in\mathcal{E}} q_e\mathbf{Q}_e$ has 0 as a simple eigenvalue with $\boldsymbol{\mu}^{eq}$ as the corresponding eigenvector, which is unique up to a scalar multiple. Let $\mathbf{G} = \sum_{e\in\mathcal{E}}\mathbf{Q}_e$. Then $\mathbf{G}$ is necessarily of the form

$$\mathbf{G} = \begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\\ \mathbf{0} & \mathbf{G}_3\end{bmatrix}, \tag{42}$$

where $\mathbf{G}_1 \in \mathbb{R}^{\bar{N}\times\bar{N}}$, $\mathbf{G}_2 \in \mathbb{R}^{\bar{N}\times(N-\bar{N})}$, $\mathbf{G}_3 \in \mathbb{R}^{(N-\bar{N})\times(N-\bar{N})}$, and $\mathbf{0}$ is the zero element of $\mathbb{R}^{(N-\bar{N})\times\bar{N}}$. If $\mathbf{G}$ does not have the block triangular structure above, then there exist indices $i \in \mathcal{V}_1$ and $j \in \mathcal{V}\backslash\mathcal{V}_1$ such that $G^{ji} > 0$. But this implies that if $\boldsymbol{\mu}^0 = \boldsymbol{\mu}^{eq}$, then $\dot{\mu}_j(0) \neq 0$ for all $j \in \mathcal{V}$, hence contradicting that $\mathbf{G}\boldsymbol{\mu}^{eq}$ is the zero element of $\mathbb{R}^N$. Moreover, since $\lim_{t\to\infty}\mu_j(t) = 0$ for all $j \in \mathcal{V}\backslash\mathcal{V}_1$ for any $\boldsymbol{\mu}^0 \in \mathbb{R}^N$, we must have that $\mathrm{spec}(\mathbf{G}_3)$ is in $\mathbb{C}_-$ and that 0 is a simple eigenvalue of $\mathbf{G}_1$. Now, let $\mathbf{a} \in \mathbf{L}^\infty(\Omega)$ be such that $a_i = 1/f_i$ if $i \in \mathcal{V}_1$ and $a_i = k_i\mathbf{1}$ if $i \in \mathcal{V}\backslash\mathcal{V}_1$ for some positive constant k_i. Then consider the operator $\boldsymbol{\mathcal{W}} = -\boldsymbol{\mathcal{M}}_\mathbf{b}\boldsymbol{\mathcal{B}}_\mathbf{a} + \sum_{e\in\mathcal{E}} q_e a_S(e)\boldsymbol{\mathcal{Q}}_e$. This operator is of the form

$$\boldsymbol{\mathcal{W}} = \begin{bmatrix}\boldsymbol{\mathcal{W}}_1 & \boldsymbol{\mathcal{W}}_2\\ \mathbf{0} & \boldsymbol{\mathcal{W}}_3\end{bmatrix}, \tag{43}$$

where $\boldsymbol{\mathcal{W}}_1 \in \mathcal{L}(X_1, X_1)$, $\boldsymbol{\mathcal{W}}_2 \in \mathcal{L}(X_1, X_2)$, $\boldsymbol{\mathcal{W}}_3 \in \mathcal{L}(X_2, X_2)$, and $\mathbf{0}$ is the zero element of $\mathcal{L}(X_2, X_1)$, with $X_1 = L^2_{a_1} \times ... \times L^2_{a_{\bar{N}}}$ and $X_2 = L^2_{a_{\bar{N}+1}} \times ... \times L^2_{a_N}$. From Proposition V.4, it follows that $\mathrm{spec}(\boldsymbol{\mathcal{W}}_3)$ lies in $\mathbb{C}_-$. Moreover, from Theorem V.1, it follows that 0 is a simple and dominant eigenvalue of $\boldsymbol{\mathcal{W}}_1$ with the corresponding eigenvector $[f_1 \;...\; f_{\bar{N}}]^T$. Then the result follows from [17][Corollary V.3.3]. □

VI. CONCLUSION

In this paper, we have proved controllability properties of a system of advection-diffusion-reaction (ADR) PDEs with zero-flux boundary condition that is defined on certain smooth domains. In contrast to previous work, we established controllability of the PDEs with bounded control inputs. Our approach to establishing controllability using spectral properties of the elliptic operators under consideration is also novel. In addition, we have provided constructive solutions to the problem of asymptotically stabilizing a class of hybrid-switching diffusion process (HSDPs) to target non-negative stationary distributions. Future work will focus on extending the arguments in this paper to the case where the corresponding HSDP has diffusion and velocity control parameters in only a small subset of the discrete behavioral states, and the diffusion coefficients are non-constant.

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