Controllability and Decentralized Stabilization of the Kolmogorov Forward Equation for Markov Chains

Karthik Elamvazhuthi [a] Shiba Biswal [b] Spring Berman [b]

[a] *Department of Mathematics, University of California, Los Angeles, CA 90095, USA (e-mail: karthikevaz@math.ucla.edu)*

[b] *School for Engineering of Matter, Transport and Energy, Arizona State University, Tempe, AZ, 85281 USA (e-mail:{sbiswal, spring.berman}@asu.edu).*

Abstract

In this paper, we provide several results on controllability and stabilizability properties of the Kolmogorov forward equation of a continuous-time Markov chain (CTMC) evolving on a finite state space, with the transition rates defined as the control parameters. First, we show that any target probability distribution can be reached asymptotically using time-varying control parameters. Second, we characterize all stationary distributions that are stabilizable using time-independent control parameters. For bidirected graphs, we construct rational and polynomial density feedback laws that stabilize stationary distributions while satisfying the additional constraint that the feedback law takes zero value at equilibrium. This last result enables the construction of decentralized density feedback controllers, using tools from linear systems theory and sum-of-squares based polynomial optimization, that stabilize a swarm of robots modeled as a CTMC to a target state distribution with no state-switching at equilibrium. In addition to these results, we prove a sufficient condition under which the classical rank conditions for controllability can be generalized to forward equations with non-negativity constraints on the control inputs. We apply this result to prove local controllability of a forward equation in which only a small subset of the transition rates are the control inputs. Lastly, we extend our feedback stabilization results to stationary distributions that have a *strongly connected support.*

Key words: Bilinear control systems; continuous-time Markov chains; controllability; swarm robotics; autonomous mobile robots;

1 INTRODUCTION

In recent years, the robotics community has devoted considerable research to the representation of robotic swarm dynamics using probability distributions and the control of their dynamics using mean-field models [16,2,12,13,7,9,10]. These representations are independent of the number of robots, and hence can be used in analysis and control synthesis approaches that are scalable with the swarm size. These results motivate a detailed analysis of control theoretic properties of Kolmogorov forward equations associated with Markov chains, with the probabilistic transition parameters as the control inputs.

In [9] we presented several results on controllability and stabilizability properties of the forward equation of a class of CTMCss, extending previous results on stabilization using open-loop control laws [1,2] and closed-loop control laws [13,18].. In this paper, we give detailed proofs of the results in [9], and we present new extensions of our results to controllability of systems with a smaller set of control variables and stabilizability of a larger class of target distributions.

The **main contributions** of this paper are the following:

(1) **Local controllability** of underactuated Kolmogorov equation. See Theorem 4.3. This result is an extension of a controllability result proved in [10] where the system was assumed to be fully-actuated.
(2) **Asymptotic controllability** of fully actuated system to target distributions . See Theorem 4.7. This result is an extension of the controllability result in [10] to target distributions that are not necessarily positive everywhere.

⋆ This paper was not presented at any IFAC meeting. This work was supported by ONR Young Investigator Award N00014-16- 1-2605 and by the Arizona State University Global Security Initiative. Corresponding author K. Elamvazhuthi.

(3) **Open-loop stabilizability** of fully actuated system to target distributions with strongly connected supports. See Proposition 5.1. This is a continuous-time version of the necessary and sufficient condition for stabilizability proved for discrete-time Markov chains in [1] and an extension of the sufficient condition presented for continuous-time [2], which was restricted to distributions that are positive everywhere.
(4) **Closed-loop stabilizability** of fully actuated system to target distributions with strongly connected supports using linear and polynomial feedback laws, with no switching of agents at equilibrium. See Theorem 5.4 and Proposition 5.5. Closed-loop feedback laws are preferable over open-loop control laws proposed in [1,2] to avoid undesirable switching of agents at equilibrium. Our stabilization results can be considered generalizations of the result in [18], where the authors constructed linear feedback laws for target distributions that are positive everywhere.

Contributions 2 and 3 were presented in [9] without proof. Contribution 4 was also partially presented in [9], without proof, for distributions that are positive everywhere. In this paper, we provide complete proofs of these results. Contribution 1 and the part of contribution 4 for general non-negative distributions are completely original to this paper.

2 NOTATION

We denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a directed graph with a set of M vertices, $\mathcal{V} = \{1, 2, ..., M\}$, and a set of $N_{\mathcal{E}}$ edges, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. An edge from vertex $i \in \mathcal{V}$ to vertex $j \in \mathcal{V}$ is denoted by $e = (i, j) \in \mathcal{E}$. We define a source map $S : \mathcal{E} \to \mathcal{V}$ and a target map $T : \mathcal{E} \to \mathcal{V}$ for which $S(e) = i$ and $T(e) = j$ whenever $e = (i, j) \in \mathcal{E}$. Throughout this paper, we will assume that the graph $\mathcal{G}$ is strongly connected. We will also assume that $(i, i) \notin \mathcal{E}$ for all $i \in \mathcal{V}$. A vector $\mathbf{x}^d \in \mathbb{R}^M$ has a *strongly connected support* if the subgraph $\mathcal{G}_{sub} = (\mathcal{V}_{sub}, \mathcal{E}_{sub})$, defined by $\mathcal{V}_{sub} = \{v \in \mathcal{V} : \mathbf{x}^d_v > 0\}$ and $\mathcal{E}_{sub} = (\mathcal{V}_{sub} \times \mathcal{V}_{sub}) \cap \mathcal{E}$, is strongly connected. Moreover, $\mathcal{V}_{sub}$ is called the *support* of the vector $\mathbf{x}^d$. For other commonly used graph-theoretical terminologies that will be used in this paper, we refer the reader to [19].

The spectrum of a matrix $\mathbf{A}$ will be denoted by spec($\mathbf{A}$). Given a vector $\mathbf{y} \in \mathbb{R}^M$, for each vertex $i \in \mathcal{V}$, the set $\sigma_{\mathbf{y}}(i) \subset \mathcal{V}$ consists of all vertices j for which there exists a directed path $\{e_k\}_{k=1}^{f}$ of some length f from j to i such that $y_{S(e_k)} = 0$ for each $k = 1, ..., f-1$. A matrix is *non-negative* if all its elements are non-negative, and it is *essentially non-negative* if all its off-diagonal elements are non-negative. A real eigenvalue λ_m of a matrix $\mathbf{A}$ will be called the *maximal eigenvalue* of $\mathbf{A}$ if $\lambda_m \geq |\lambda|$ for all $\lambda \in$ spec($\mathbf{A}$). We will denote the *conical span* of a set C of m vectors $\mathbf{x}_i \in \mathbb{R}^M$, $i = 1, ..., m$, by co span$(C) = \{\sum_{i=1}^{m} \alpha_i \mathbf{x}_i : \mathbf{x}_i \in C,\ \alpha_i \in \mathbb{R}_{\geq 0},\ i = 1, ..., m\}$.

The matrix $\mathcal{L}_{out}(\mathcal{G}) = \mathbf{D}_{out}(\mathcal{G}) - \mathbf{A}(\mathcal{G}) \in \mathbb{R}^{M \times M}$ denotes the *out-Laplacian* of the graph $\mathcal{G}$, where $\mathbf{D}_{out}(\mathcal{G})$ is the out-degree matrix of $\mathcal{G}$ and $\mathbf{A}(\mathcal{G})$ is the adjacency matrix of $\mathcal{G}$. $\mathbf{D}_{out}(\mathcal{G})$ is a diagonal matrix for which $(\mathbf{D}_{out}(\mathcal{G}))^{ii}$ is the total number of edges e such that $S(e) = i$. The entries of $\mathbf{A}(\mathcal{G})$ are defined as $(\mathbf{A}(\mathcal{G}))^{ij} = 1$ if $(j, i) \in \mathcal{E}$, and 0 otherwise. When $\mathcal{G}$ is bidirected, $\mathcal{L}_{out}(\mathcal{G})$ is the usual Laplacian of the graph, and we will drop the subscript and denote it by $\mathcal{L}(\mathcal{G})$. For a subset $B \subset \mathbb{R}^M$, int B and Bd B will refer to the interior and the boundary, respectively, of B.

3 CONTROL MODEL

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with the set of vertices $\mathcal{V}$ and edges $\mathcal{E}$. We consider the following control system,

$$\dot{\mathbf{x}}(t) = \sum_{e \in \mathcal{E}} u_e(t) \mathbf{B}_e \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}^0 \in \mathcal{P}(\mathcal{V}), \tag{1}$$

where $\mathbf{B}_e$ are control matrices whose entries are given by

$$B_e^{ij} = \begin{cases} -1 & \text{if } i = j = S(e), \\ 1 & \text{if } i = T(e),\ j = S(e), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The focus of this paper is to study **controllability** and **stabilizability** properties of the control system (1). We first consider the case where only a subset of the transition rates of this system can be specified, i.e., the control system is **underactuated**. Then we consider different types of stabilizability properties of the system for the fully actuated case. The analysis of this control system is motivated by potential applications in control of robotic swarms where $\mathcal{V}$ denotes the state-space of the robots, $x_v(t)$ denotes the density of robots on the set $v \in \mathcal{V}$ and $u_e(t)h$ denotes the probability of a robot transitioning from the source $S(e)$ of the edge $e \in \mathcal{E}$ to the target edge $T(e)$, at time $t+h$ for h infinitesimally small. For a detailed discussion on the correspondence between this control system and the stochastic processes associated with the agent dynamics we refer the reader to [8].

Remark 3.1 *We note that $\mathcal{P}(\mathcal{V})$ is an invariant set for system* (1) *because $\mathbf{B}_e$ has off-diagonal positive entries, its columns sum to 0, and the control inputs $u_e(t)$ are constrained to be non-negative. This fact will be used throughout the paper.*

4 CONTROLLABILITY ANALYSIS

4.1 Local Controllability of the Underactuated System

In this section, we prove a result (Theorem 4.3) on the controllability of system (1). Due to the non-negativity constraints on the control inputs, this result cannot be directly concluded from classical tests of controllability such as the Kalman rank condition or the Lie algebra rank condition [3]. Here, we first state a general result which implies that these classical rank conditions for controllability have a straightforward generalization to systems with non-negative control inputs. These generalized rank conditions can be used to establish the controllability result for system (1) that we proved in [10] for the case where all control inputs can be specified. The following result that we present has already been established in [15] for controllable nonlinear control systems with positivity constraints on the control inputs in the case where the linearized control system with the same constraints is also controllable. We state it here it here in this form, for the reader's convenience, as it will be used later to prove our result on local controllability.

Theorem 4.1 *Consider the control-affine system*

$$\dot{\mathbf{x}}(t) = \mathbf{f}_0(\mathbf{x}(t)) + \sum_{i=1}^{N} u_i(t)\mathbf{f}_i(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}^0 \tag{3}$$

with smooth vector fields $\mathbf{f}_i : \mathbb{R}^M \rightarrow \mathbb{R}^M$ *for* $i = 0, ..., N$. *Suppose that* $\mathbf{x}^f \in \mathbb{R}^M$ *and there exist measurable control inputs* $u_i : [0, T] \rightarrow \mathbb{R}$, $i = 1, ..., N$, *such that a unique solution of the system* (3) *exists and satisfies* $\mathbf{x}(T) = \mathbf{x}^f$. *Additionally, suppose that the following condition holds for all* $t \in [0, T]$*:*

$$\begin{aligned} \text{span}\{\mathbf{f}_i(\mathbf{x}(t)) :\ & i = 1, ..., N\} \\ &= \text{co span}\{\mathbf{f}_i(\mathbf{x}(t)) : i = 1, ..., N\}. \end{aligned} \tag{4}$$

Then there exist measurable non-negative control inputs $\tilde{u}_i : [0, T] \rightarrow \mathbb{R}_{\geq 0}$, $i = 1, ..., N$, *such that the state* $\mathbf{x}(t)$ *evolves according to the following system for almost every (a.e.)* $t \in [0, T]$*:*

$$\dot{\mathbf{x}}(t) = \mathbf{f}_0(\mathbf{x}(t)) + \sum_{i=1}^{N} \tilde{u}_i(t)\mathbf{f}_i(\mathbf{x}(t)), \ \mathbf{x}(0) = \mathbf{x}^0 \tag{5}$$

These generalized rank conditions can be used to establish the following controllability result for system (1) that we proved in [10], in the case where all control inputs can be specified. Note that the result in [10] is a stronger global controllability result than the local controllability result that we will be proving in the following theorem. It was possible to establish globabl controllability of the system in [10] because the system was assumed to be fully actuated.

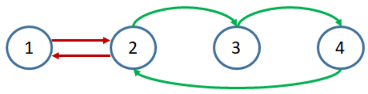

Fig. 1. Graph $\mathcal{G}$ in Example 4.2. Edges in $\mathcal{E}_0$ are uncontrolled and denoted by red arrows. Edges in $\mathcal{E}_1$ are controlled and denoted by green arrows.

The following example demonstrates the possibility of achieving local controllability of system (1) even when the system is underactuated, in contrast with the fully actuated system in [10] [Theorem IV.17].

Example 4.2 *Let* $\mathcal{V} = \{1, 2, 3, 4\}$, $\mathcal{E}_0 = \{(1, 2), (2, 1)\}$, *and* $\mathcal{E}_1 = \{(2, 3), (3, 4), (4, 2)\}$. *We set* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, *where* $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$ *(see Fig. 1). We consider a variant of the control system* (1) *in which the control inputs* $u_e(t)$, $e \in \mathcal{E}_0$ *are each set to* 1, *and the inputs* $u_e(t) \in \mathbb{R}$, $e \in \mathcal{E}_1$ *can be designed:*

$$\dot{\mathbf{x}}(t) = \sum_{e \in \mathcal{E}_0} \mathbf{B}_e \mathbf{x}(t) + \sum_{e \in \mathcal{E}_1} u_e(t) \mathbf{B}_e \mathbf{x}(t), \ \mathbf{x}(0) = \mathbf{x}^0 \in \mathcal{P}(\mathcal{V}). \tag{6}$$

Let $\mathbf{x}^d = [\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4}]^T \in \mathcal{P}(\mathcal{V})$. *The Kalman rank condition can be used to verify that the control system* (6) *linearized about the point* $\mathbf{x}^d$ *is controllable. Hence, system* (6) *is locally controllable [3]; that is, given* $T > 0$, *there exists* $r > 0$ *and a neighborhood* $B(\mathbf{x}^d, r) \cap \mathcal{P}(\mathcal{V})$ *of* $\mathbf{x}^d$ *such that for each* $\mathbf{x}^d \in B(\mathbf{x}^d, r) \cap \mathcal{P}(\mathcal{V})$, *there exist measurable control inputs* $u_e(t)$ *for* $e \in \mathcal{E}_1$, *possibly with negative values at some time* t, *such that* $\mathbf{x}(T) = \mathbf{x}^d$. *Moreover, a straightforward computation of* $\mathbf{B}_e\mathbf{y}$ *confirms that* $\text{span}\{\mathbf{B}_e\mathbf{y} : e \in \mathcal{E}_1\} = \text{co span}\{\mathbf{B}_e\mathbf{y} : e \in \mathcal{E}_1\}$ *for all* $\mathbf{y} \in \text{int } \mathcal{P}(\mathcal{V})$. *Hence, by Theorem 4.1, system* (6) *is locally controllable at* $\mathbf{x}^d$ *using a set of non-negative control inputs corresponding to the edges in* $\mathcal{E}_1$.

Example 4.2 can be generalized to give the following sufficient condition for local controllability.

Theorem 4.3 *Let* $\mathcal{G}$ *be a strongly connected graph with* $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$. *In the control system* (6) *with this graph, the control inputs* $u_e(t)$, $e \in \mathcal{E}_1$ *may be negative. Let system* (6) *be small-time locally controllable at* $\mathbf{x}^d \in \text{int } \mathcal{P}(\mathcal{V})$. *Then the bilinear system* (1) *is small-time locally controllable at* $\mathbf{x}^d \in \text{int } \mathcal{P}(\mathcal{V})$ *if* $e \in \mathcal{E}_1$ *implies that there exists a directed path* $(e_i)_{i=1}^p$ *of length* p *from vertex* $T(e)$ *to vertex* $S(e)$ *such that* $e_i \in \mathcal{E}_1$ *for all* $i \in \{1, ..., p\}$. *In particular, with this assumption on* $\mathcal{E}_1$, *if the linearization of the control system* (6) *is controllable, then system* (6) *is small-time locally controllable using non-negative control inputs.*

The controllability result in proved in [10] for the case where all control inputs can be specified, in contrast to the result in Theorem 4.3, which applies when only a subset of these inputs can be designed. To prove the Theorem on global controllability the result in [10], it was sufficient to assume that the graph $\mathcal{G}$ is strongly

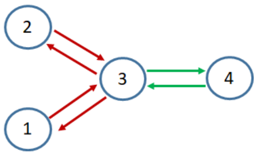

Fig. 2. Graph $\mathcal{G}$ in Example 4.4. Edges in $\mathcal{E}_0$ are denoted by red arrows. Edges in $\mathcal{E}_1$ are denoted by green arrows.

connected. The following example shows that when only a subset of the control inputs can be designed, strong connectivity of the graph is not a sufficient condition for proving local controllability of system (1).

Example 4.4 *Let $\mathcal{V} = \{1,2,3,4\}$, $\mathcal{E}_0 = \{(1,3),(3,1),(2,3),(3,2)\}$, and $\mathcal{E}_1 = \{(3,4),(4,3)\}$. We set $\mathcal{G} = (\mathcal{V},\mathcal{E})$, where $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$ (see Fig. 2). Note that $\mathcal{G}$ is strongly connected. We consider the control system* (6) *with this graph. If $\mathbf{x}^0 \in \mathcal{P}(\mathcal{V})$ is such that $x_1^0 = x_2^0$, then the solution $\mathbf{x}(t)$ of system* (6) *satisfies $x_1(t) = x_2(t)$ for all $t \geq 0$ for any choice of control inputs $u_e(t) \in \mathbb{R}$, $e \in \mathcal{E}_1$. Hence, although $\mathcal{G}$ is strongly connected, system* (6) *is not locally controllable at any point $\mathbf{x}^d \in \mathcal{P}(\mathcal{V})$ that satisfies $x_1^d = x_2^d$. The nature of this obstruction to controllability is similar to the one in leader-based control of linear consensus protocols [19], in which inputs act at the vertices of the graph rather than the edges, and symmetries in the graph with respect to input locations have a detrimental effect on the controllability of the system.*

As we demonstrated with a simple counterexample in [9], distributions that correspond to points on the boundary of $\mathcal{P}(\mathcal{V})$ might not be reachable in finite time by solutions of the control system (1), even with the use of unbounded control inputs. In the following theorem, we prove a general negative controllability result for the case where the control inputs are constrained to be bounded.

Proposition 4.5 *Let $\mathbf{x}^0 \in \mathcal{P}(\mathcal{V})$ be such that $x_i^0 > 0$ for some $i \in \mathcal{V}$ and let $T > 0$. Suppose that the control inputs $u_e(t)$ are essentially bounded over the time interval $[0,T]$. Then the solution $\mathbf{x}(t)$ of the control system* (1) *satisfies $x_i(t) > 0$ for all $t \in [0,T]$.*

Proof. For the sake of contradiction, suppose that there exist bounded piecewise control inputs $u_e(t)$ such that the solution $\mathbf{x}(t)$ of the control system (1) satisfies $x_i(T) = 0$. Note that $x_i(t) = x_i^0 + \sum_{e\in\mathcal{E}_a} \int_0^t u_e(\tau)x_{S(e)}(\tau)d\tau - \sum_{e\in\mathcal{E}_b} \int_0^t u_e(\tau)x_i(\tau)d\tau$ for all $t \in [0,T]$, where $\mathcal{E}_a$ is the set of edges e such that $T(e) = i$ and $\mathcal{E}_b$ is the set of edges e such that $S(e) = i$. Then $x_i(t) \geq \hat{x}_i(t) := \exp\left(-\sum_{e\in\mathcal{E}_b} \int_0^t \|u_e\|_\infty d\tau\right)x_i^0 = x_i^0 - \sum_{e\in\mathcal{E}_b} \int_0^t \|u_e\|_\infty \hat{x}_i(\tau)d\tau$ for all $t \in [0,T]$. Otherwise, due to continuity of the solution $\mathbf{x}(t)$ with respect to time t, there would exist a time $t_{in} \in (0,T)$ at which $\dot{x}_i(t_{in}) < \dot{\hat{x}}_i(t_{in})$ and $x_i(t_{in}) = \hat{x}_i(t_{in})$. Hence, the inequality $x_i(t) \geq \hat{x}_i(t)$ must hold for all $t \in [0,T]$. However, given the initial assumption that $x_i(T) = 0$, this inequality leads to a contradiction since $\exp\left(-\sum_{e\in\mathcal{E}_b} \int_0^t \|u_e\|_\infty d\tau\right)x_i^0 > 0$. Therefore, the boundary set of elements of $\mathcal{P}(\mathcal{V})$ with $y_i = 0$ is not reachable in finite time with the use of piecewise constant control inputs that are bounded from above by $\max_{e\in\mathcal{E}} \|u_e\|_\infty$ and bounded from below by $-\max_{e\in\mathcal{E}} \|u_e\|_\infty$. Since any essentially bounded function can be approximated using piecewise constant functions (in a suitable weak topology), this implies that the set of elements $\mathbf{y}$ in $\mathcal{P}(\mathcal{V})$ with $y_i = 0$ is not reachable in finite time using essentially bounded control inputs. Thus, there is a positive uniform lower bound on each $x_i(T)$, and therefore it is not possible to construct a sequence of uniformly bounded control inputs that drive a state $x_i(t)$ to 0 at time T. $\square$

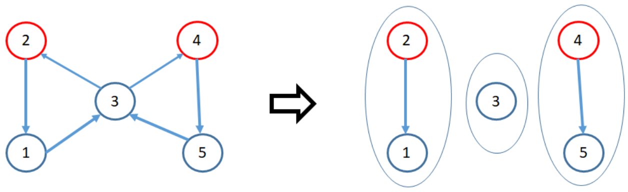

Fig. 3. Illustrative example of partitioning a graph $\mathcal{G}$ in the proof of Theorem 4.7. The graph on the left is the original graph $\mathcal{G}$, in which vertex i is red if $x_i^d = 0$ and blue if $x_i^d > 0$. The graphs on the right show the partition of $\mathcal{G}$ into $\mathcal{R} = 3$ disjoint graphs $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$, each of which is a rooted in-branching with a root vertex i for which $x_i^d > 0$.

When performing controllability analysis of bilinear systems on vector spaces, it is common to instead study the controllability of a related control system on the set of matrices. For instance, consider the following control system:

$$\dot{\mathbf{X}}(t) = \sum_{e\in\mathcal{E}} u_e(t)\mathbf{B}_e\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{I} \tag{7}$$

Controllability of system (7) implies controllability of system (1). Specifically, if system (7) is controllable on the set of non-negative matrices that keep $\mathcal{P}(\mathcal{V})$ invariant, then system (1) is controllable on $\mathcal{P}(\mathcal{V})$. This follows from the observation that, if $u_e(t)$ is a given set of control inputs, then $\mathbf{X}(t)\mathbf{x}_0 = \mathbf{x}(t)$ is the solution of system (1). This form of *lifting* of system (1) to (7) could be advantageous from a mathematical point of view due to the fact that the set of matrices with zero column sums form a *Lie group*. Hence, one can potentially apply some of the extensive number of results on controllability properties of bilinear systems on Lie groups [4,11] to study system (1). In fact, using such Lie group-theoretic techniques, in a recent work [17], controllability system (7) has been established for the case when the graph $\mathcal{G}$ is bidirected and the controls $u_e(t)$ are allowed to be negative. For applicability of this controllability result in [17] to stochastic processes, it is of interest to understand if controllability is preserved when non-negativity constraints are imposed on the control inputs. However, using the result in Theorem 4.5 we can see that, with non-negativity constraints on $u_e(t)$, system (7) is not controllable on the set of non-negative matrices that preserve $\mathcal{P}(\mathcal{V})$. For example, let $\mathbf{P}$ be the matrix defined by $P^{ij} = 1$ if $i = 1$ and 0 otherwise. Then $\mathbf{P}$ is a matrix with non-negative elements and $\mathbf{1}^T\mathbf{P} = \mathbf{1}^T$, where $\mathbf{1} \in \mathbb{R}^M$ is vector with all ones. Hence, $\mathbf{P}$ preserves $\mathcal{P}(\mathcal{V})$. For any $\mathbf{x} \in \mathcal{P}(\mathcal{V})$, $\mathbf{Px} = \mathbf{y} \in \mathcal{P}(\mathcal{V})$ satisfies $y_1 = 1$ and therefore, $y_i = 0$ for all $i \neq 1$. However, if the solution $\mathbf{X}(t)$ satisfies $\mathbf{X}(T) = \mathbf{P}$ for some essentially bounded non-negative control inputs $u_e(t)$, then we would arrive at a contradiction with Theorem 4.5 since we cannot have that $x_2(T) = (\mathbf{X}(T)\mathbf{x}^0)_2 = 0$ if $x_2^0 > 0$. Therefore, we have the following result.

Proposition 4.6 *Let $\mathcal{SM}$ be the set of non-negative matrices that keep $\mathcal{P}(\mathcal{V})$ invariant. Then there exist points in $\mathcal{SM}$ that are not reachable using essentially-bounded control inputs $u_e(t)$ that are non-negative.*

Note that the above result is not due to the failure of the Lie algebra rank condition [3]. In fact, it can be shown that Lie algebra generated by the matrices $\mathbf{B}_e$ has full rank, and hence system (7) has the *accessibility* property. See [9] for the relevant computations.

4.2 Asymptotic Controllability

In contrast with the result stated in Proposition 4.5, which shows that the boundary points of $\mathcal{P}(\mathcal{V})$ are not reachable in finite time, the next theorem proves that these points can be reached asymptotically as $t \to \infty$.

Theorem 4.7 *Suppose that $\mathbf{x}^0 \in \mathcal{P}(\mathcal{V})$ is the initial distribution, and $\mathbf{x}^d \in \mathcal{P}(\mathcal{V})$ is the desired distribution. Then for each $e \in \mathcal{E}$, there exists a set of time-dependent control inputs $u_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $e \in \mathcal{E}$, such that the solution $\mathbf{x}(t)$ of the control system (1) satisfies $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^d$.*

Proof. We define the set $\mathcal{R} = \{i : \ x_i^d > 0, \ \ i = 1, ..., M\}$ with cardinality $N_\mathcal{R}$. Let $\mathcal{I} : \{1, 2, ..., N_\mathcal{R}\} \to \mathcal{R}$ be a bijective map that defines an ordering on $\mathcal{R}$. Then we recursively define a collection $\{\mathcal{V}_n\}$ of disjoint subsets of $\mathcal{V}$ as follows:

$$\begin{aligned}\mathcal{V}_1 &= \{\mathcal{I}(1)\} \cup \{i \in \mathcal{V} : \ x_i^d = 0 \ \ s.t. \ \ i \in \sigma_{\mathbf{x}^d}(\mathcal{I}(1))\} \\ \mathcal{V}_n &= \{\mathcal{I}(n)\} \cup \{i \in \mathcal{V} : \ x_i^d = 0 \ \ s.t. \ \ i \in \sigma_{\mathbf{x}^d}(\mathcal{I}(n)) \\ &\qquad\qquad\qquad\qquad \text{and} \ \ i \notin \cup_{k=1}^{n-1}\mathcal{V}_k\}\end{aligned}$$

for each $n \in \{2, 3, ..., N_\mathcal{R}\}$. We note that $\mathcal{V} = \cup_{n=1}^{N_\mathcal{R}} \mathcal{V}_n$. Let $\mathbf{x}^{in} \in \text{int}(\mathcal{P}(\mathcal{V}))$ be some element such that $\sum_{k\in\mathcal{V}_n} x_k^{in} = x_{\mathcal{I}(n)}^d$ for each $n \in \{1, 2, ..., N_\mathcal{R}\}$. From the global controllability result stated [10][Theorem IV.7], we know that there exists a control input $u_e^1 : [0, T] \to \mathbb{R}_{\geq 0}$ for each $e \in \mathcal{E}$ such that the solution $\mathbf{x}(t)$ of system (1) satisfies $\mathbf{x}(T) = \mathbf{x}^{in}$. Now we will design $\{u_e\}_{e\in\mathcal{E}}$ such that $u_e(t) = u_e^1(t)$ for each $t \in [0, T]$ and $u_e(t) = a_e$ for each $t \in (T, \infty]$, where a_e is defined as follows:

$$a_e = \begin{cases} 0 & \text{if } S(e) \in \mathcal{V}_n \text{ and } T(e) \notin \mathcal{V}_n, \quad n \in \{1, ..., N_\mathcal{R}\}, \\ 0 & \text{if } S(e) = \mathcal{I}(n) \ \text{ for some } n \in \{1, ..., N_\mathcal{R}\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then the solution of system (1) for $t > T$ can be constructed from the solution of the following decoupled set of ODEs:

$$\dot{\mathbf{y}}_n(t) = -\mathcal{L}_{out}(\mathcal{G}_n)\mathbf{y}_n(t), \quad \mathbf{y}_n(T) = \mathbf{y}_n^0 \in \mathcal{P}(\mathcal{V}_n) \qquad (8)$$

for $n = 1, ..., N_\mathcal{R}$. Here, $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ for each $n \in \{1, ..., N_\mathcal{R}\}$, where $e \in \mathcal{E}_n$ if $S(e), T(e) \in \mathcal{V}_n$ and $a_e = 1$. An illustration of the construction of the graphs $\mathcal{G}_n$ is shown in Fig. 4.1.

The solution of system (8) is related to the solution of system (1) with $\mathbf{x}(T) = \mathbf{x}^{in}$ through a suitable permutation matrix $\mathbf{P}$, defined such that $\mathbf{Px}(t) = [\mathbf{y}_1(t) \ \mathbf{y}_2(t) \ ... \ \mathbf{y}_{N_\mathcal{R}}(t)]$. Since each graph $\mathcal{G}_n$ is a rooted in-branching subgraph, the process generated by $-\mathcal{L}_{out}(\mathcal{G}_n)^T$ has a unique stationary distribution [5][Proposition 10]. Moreover, by construction, this unique, globally stable stationary distribution is the vector $[x_{\mathcal{I}(n)}^d \ \ \mathbf{0}_{1\times(|\mathcal{V}_n|-1)}]^T$, where $|\mathcal{V}_n|$ is the cardinality of the set $\mathcal{V}_n$. This implies that $\lim_{t\to\infty} \mathbf{P}^{-1}\mathbf{y}(t) = \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^d$. By concatenating the control inputs $\{u_e^1\}_{e\in\mathcal{E}}$ and $\{a_e\}_{e\in\mathcal{E}}$, we obtain the desired asymptotic controllability result. $\square$

The significance of Theorem 4.7 is that for any given target distribution $\mathbf{x}^d \in \mathcal{P}(\mathcal{V})$, we can now conclude there exists a globally asymptotically stabilizing controller that stabilizes the closed-loop system (9) $\mathbf{x}^d$. This follows from an important result in the literature on the relationship between asymptotic controllability and feedback stabilizability [6]. In subsequent sections, we explicitly construct globally asymptotically stabilizing controllers that can only stabilize $\mathbf{x}^d \in \mathcal{P}(\mathcal{V})$ with strongly connected support, a more restricted class of target distributions, but that can be chosen to have a decentralized structure. The existence of globally asymptotically stabilizing controllers with such a structure does not follow from the results in [6].

5 Stabilizability Analysis

5.1 Open-loop Stabilization

In this section, we establish stabilizability of the system (1) using constant control inputs that are independent of the state of each agent and time. From a practical point of view, these are the simplest type of control laws that can be implemented in practice as each agent needs to be cognizant only of its own state in order to be able to control such a control law. We note that the corresponding result for agents evolving according to discrete-time Markov chains has been proved in [1].

Proposition 5.1 *Let $\mathcal{G}$ be a strongly connected graph. Suppose that $\mathbf{x}^0 \in \mathcal{P}(\mathcal{V})$ is an initial distribution and $\mathbf{x}^d \in \mathcal{P}(\mathcal{V})$ is a target distribution. Additionally, assume that $\mathbf{x}^d$ has strongly connected support. Then there is a set of parameters, $a_e \in [0,\infty)$ for each $e \in \mathcal{E}$, such that if $u_e(t) = a_e$ for all $t \in [0,\infty)$ and for each $e \in \mathcal{E}$ in system (1), then the solution $\mathbf{x}(t)$ of this system satisfies $\|\mathbf{x}(t) - \mathbf{x}^d\| \le Me^{-\lambda t}$ for all $t \in [0,\infty)$ and for some positive parameters M and λ that are independent of $\mathbf{x}^0$.*

Remark 5.2 ***(Non-applicability of the Perron-Frobenius theorem)*** *Before we prove Proposition 5.1, we note that since it considers non-negative target distributions $\mathbf{x}^d$, it cannot be concluded from the Perron-Frobenius theorem, which only applies to positive $\mathbf{x}^d$.*

Proof. Let $\mathcal{V}_s \subset \mathcal{V}$ be the support of $\mathbf{x}^d$. From this vertex set, we construct a new graph $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$, where $e = (i,j) \in \mathcal{E}$ implies that $e \in \tilde{\mathcal{E}}$ if and only if $i \in \mathcal{V}_s$ implies that $j \notin \mathcal{V} \backslash \mathcal{V}_s$. Then it follows from [5][Proposition 10] that the process generated by the transition rate matrix $-\mathcal{L}_{out}(\tilde{\mathcal{G}})^T$ has a unique, globally stable invariant distribution if we can establish that $\tilde{\mathcal{G}}$ has a *rooted in-branching* subgraph. This implies that $\tilde{\mathcal{G}}$ must have a subgraph $\tilde{\mathcal{G}}_{sub} = (\mathcal{V}, \mathcal{E}_{sub})$ which has no directed cycles and for which there exists a root node, v_r, such that for every $v \in \mathcal{V}$ there exists a directed path from v to v_r. This is indeed true for the graph $\tilde{\mathcal{G}}$, which can be shown as follows. First, let $r \in \mathcal{V}$ such that $x^d_r > 0$. From the assumption that $\mathcal{G}$ is strongly connected and the construction of $\tilde{\mathcal{G}}$, it can be concluded that there exists a directed path in $\tilde{\mathcal{E}}$ from any $v \in \mathcal{V}$ to r. Now, for each $n \in \mathbb{Z}_+$, the set of positive integers, let $\mathcal{N}_n(r)$ be the set of all vertices for which there exists a directed path of length n to r. For each $n > 1$, let $\tilde{\mathcal{N}}_n(r) = \mathcal{N}_n(r) \backslash \cup_{m=1}^{n-1} \mathcal{N}_m(r)$. We define $\mathcal{E}_{sub}$ by setting $e \in \mathcal{E}_{sub}$ if and only if $e \in \tilde{\mathcal{E}}$, $S(e) \in \tilde{\mathcal{N}}_n(r)$, and $T(e) \in \tilde{\mathcal{N}}_{n-1}(r)$ for some $n > 1$. Then $\tilde{\mathcal{G}}_{sub} = (\mathcal{V}, \mathcal{E}_{sub})$ is the desired rooted in-branching subgraph.

The matrix $-\mathcal{L}_{out}(\tilde{\mathcal{G}})^T$ is the generator of a CTMC, since $\mathcal{L}_{out}(\tilde{\mathcal{G}})^T \mathbf{1} = \mathbf{0}$ and its off-diagonal entries are positive. Moreover, as we have shown, $\tilde{\mathcal{G}}$ has a rooted in-branching subgraph. Hence, there exists a unique vector $\mathbf{z}$ such that $-\mathcal{L}_{out}(\tilde{\mathcal{G}})\mathbf{z} = \mathbf{0}$ and $\mathbf{z} \in \mathcal{P}(\mathcal{V})$. The vector $\mathbf{z}$ is nonzero only on $\mathcal{V}_s$, since the subgraph corresponding to $\mathcal{V}_s$ is strongly connected. Then we consider a positive definite diagonal matrix $\mathbf{D} \in \mathbb{R}^{M\times M}$ such that $D^{ii} = z_i/x^d_i$ if $i \in \mathcal{V}_s$ and is an arbitrary strictly positive value for any other $i \in \mathcal{V}$. The matrix $-\mathbf{D}\mathcal{L}_{out}(\tilde{\mathcal{G}})^T$ is also the generator of a CTMC. Moreover, $\mathbf{x}^d$ is the unique stationary distribution of the process generated by $-\mathbf{D}\mathcal{L}_{out}(\tilde{\mathcal{G}})^T$, since $\mathbf{x}^d$ lies in the null space of $\mathbf{G} = -\mathcal{L}_{out}(\tilde{\mathcal{G}})\mathbf{D}$ by construction $\big(\mathcal{L}_{out}(\tilde{\mathcal{G}})\mathbf{D}\mathbf{x}^d = \mathcal{L}_{out}(\tilde{\mathcal{G}})\mathbf{z} = \mathbf{0}\big)$. The simplicity of the principal eigenvalue at 0 for the matrix $-\mathbf{D}\mathcal{L}_{out}(\tilde{\mathcal{G}})^T$ is inherited by the same eigenvalue of the matrix $\mathbf{G}$. Then the result follows by setting $a_e = G^{T(e)S(e)}$ for each $e \in \mathcal{E}$ and by noting that since $\mathbf{G}^T$ is the generator of a CTMC, and its eigenvalue at 0 has the aforementioned properties and is simple, then the rest of the spectrum of $\mathbf{G}$ lies in the open left half of the complex plane. In particular, for this choice of a_e, the right-hand side of system (1) becomes $\sum_{e\in\mathcal{E}} a_e \mathbf{B}_e \mathbf{x}(t) = \mathbf{G}\mathbf{x}(t)$, and therefore the solution $\mathbf{x}(t)$ of (1) exponentially converges to $\mathbf{x}^d$. □

5.2 Closed-loop Stabilization

Now we investigate the stabilizability properties of the system (1) using *closed-loop feedback laws*. While open-loop time-independent control laws are require lesser amount of information than closed-loop feedback laws, open-loop time-independent control laws have the disadvantage that the control inputs are non-zero at equilibrium. To address this specific issue we will be looking for feedback control laws that take zero value at equilibrium, and hence reduce the amount of switching that agents need to be as $t \to \infty$. Particularly, our goal will be to construct *decentralized feedback laws* $k_e : \mathbb{R}_{\ge 0}$ such that $k(x^d_i) = 0$ and a given target equilibrium distribution $\mathbf{x}_d \in \mathcal{P}(\mathcal{V})$ is globally asymptotically stable for the following system,

$$\dot{\mathbf{x}}(t) = \sum_{e\in\mathcal{E}} k_e(\mathbf{x}(t))\mathbf{B}_e\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}^0 \in \mathcal{P}(\mathcal{V}). \tag{9}$$

Note that stabilizability using centralized feedback follows from the controllability result in [10][Theorem IV.17] and from general control theoretic results that relate controllability and stabilizability [21,6]. Hence, our focus in this section is to establish stabilizability using decentralized control laws.

Lemma 5.3 *Define $\mathbf{x}^d \in \mathrm{int}(\mathcal{P}(\mathcal{V}))$. For each $e \in \mathcal{E}$ and each $\mathbf{y} \in \mathbb{R}^M$, let $k_e : \mathbb{R}^M \to (-\infty,\infty)$ be given by*

$$k_e(\mathbf{y}) = x^d_{T(e)} y_{S(e)} - x^d_{S(e)} y_{T(e)} \tag{10}$$

in system (9). Then, $\mathbf{x}^d$ is locally exponentially stable on the space $\mathcal{P}(\mathcal{V})$. That is, there exists $r > 0$ such that $\|\mathbf{x}^0 - \mathbf{x}^d\|_2 < r$ and $\mathbf{x}^0 \in \mathcal{P}(\mathcal{V})$ imply that the solution $\mathbf{x}(t)$ of system (9) satisfies the inequality: $\|\mathbf{x}(t) - \mathbf{x}^d\|_2 \le M_0 e^{-\lambda t}$, for all $t \in [0,\infty)$ and for some parameters $M_0 >$

0 *and* $\lambda > 0$ *that depend only on* r*. If* $\mathcal{G}$ *is bidirected, then* $\mathbf{x}^d$ *is also asymptotically stable.*

Proof. We use the linearization of system (9) about $\mathbf{x}^d$ to establish local exponential stability. Consider the vector field $\mathbf{f}^e = [f_1^e \; f_2^e \; ... \; f_M^e]^T$ given by

$$f_i^e(\mathbf{y}) = \begin{cases} -(x^d_{T(e)} y_{S(e)} - x^d_{S(e)} y_{T(e)}) y_{S(e)} & \text{if } i = S(e), \\ (x^d_{T(e)} y_{S(e)} - x^d_{S(e)} y_{T(e)}) y_{S(e)} & \text{if } i = T(e), \\ 0 & \text{otherwise} \end{cases}$$

for each $\mathbf{y} \in \mathbb{R}^M$. Then for each $e \in \mathcal{E}$, we define the matrix $\mathbf{A}_e \in \mathbb{R}^M \times \mathbb{R}^M$ as follows:

$$A_e^{ij} = \begin{cases} \frac{\partial f^e_{S(e)}}{\partial y_{S(e)}}\Big|_{\mathbf{y}=\mathbf{x}^d} = -x^d_{T(e)} x^d_{S(e)} & \text{if } i = j = S(e), \\ \frac{\partial f^e_{S(e)}}{\partial y_{T(e)}}\Big|_{\mathbf{y}=\mathbf{x}^d} = (x^d_{S(e)})^2 & \text{if } i = S(e), j = T(e), \\ \frac{\partial f^e_{T(e)}}{\partial y_{T(e)}}\Big|_{\mathbf{y}=\mathbf{x}^d} = -(x^d_{S(e)})^2 & \text{if } i = j = T(e), \\ \frac{\partial f^e_{T(e)}}{\partial y_{S(e)}}\Big|_{\mathbf{y}=\mathbf{x}^d} = x^d_{T(e)} x^d_{S(e)} & \text{if } i = T(e), j = S(e), \\ 0 & \text{otherwise.} \end{cases}$$

Now we define the matrix $\mathbf{G} \in \mathbb{R}^{M \times M}$ as $\mathbf{G} = \sum_{e \in \mathcal{E}} \mathbf{A}_e$. Note that $G^{S(e)T(e)} > 0$ for each $e \in \mathcal{E}$, since $\mathbf{x}^d \in \text{int}(\mathcal{P}(\mathcal{V}))$. Moreover, $\mathbf{1}^T\mathbf{G} = \mathbf{0}$, and the off-diagonal terms of $\mathbf{G}$ are positive. Hence, $\mathbf{G}$ is an irreducible transition rate matrix. It is a classical result that this implies that $\mathbf{G}$ has its principal eigenvalue at 0, which is simple. The other eigenvalues of $\mathbf{G}$ lie in the open left half of the complex plane. However, note that the equilibrium point $\mathbf{x}^d$ is *non-hyperbolic*, since the principal eigenvalue of $\mathbf{G}$ is at 0. Hence, local exponential stability of the original nonlinear system does not immediately follow. However, it follows that there exists an $(M-1)-$dimensional local stable manifold of the system that is tangential to $\mathcal{P}(\mathcal{V})$ at $\mathbf{x}^d \in \mathcal{P}(\mathcal{V})$. Noting that the set $\{\mathbf{y} \in \mathbb{R}^M; \sum_{i=1}^M y_i = c\}$ is invariant for solutions of the system (9) for any $c \in \mathbb{R}$, it follows that the stable manifold is in fact in $\mathcal{P}(\mathcal{V})$. From this, the result follows.

To prove asymptotic stability of $\mathbf{x}^d$ for bidirected graphs, consider the continuously differentiable function $V : \mathbb{R}^M \to \mathbb{R}_{\geq 0}$ given by

$$V(\mathbf{y}) = \frac{1}{2}(\mathbf{y} - \mathbf{x}^d)^T \mathbf{D}(\mathbf{y} - \mathbf{x}^d) \tag{11}$$

for all $\mathbf{y} \in \mathbb{R}^M$, where $\mathbf{D} \in \mathbb{R}^{M \times M}$ is defined as $\mathbf{D} = [\text{diag}(\mathbf{x}^d)]^{-1}$. Then

$$\dot{V}(\mathbf{x}(t)) = \sum_{e \in \mathcal{E}} x_{S(e)}(t)(x^d_{T(e)} x_{S(e)}(t) - x^d_{S(e)} x_{T(e)}(t))^2.$$

Thus, $\dot{V}(\mathbf{x}(t)) \leq 0$ for all $t \in [0, \infty)$, with the equality $\dot{V}(\mathbf{x}(t)) = 0$ holding only when $\mathbf{x}(t) = \mathbf{x}^d$. Then, the asymptotic stability of $\mathbf{x}^d$ follows from *LaSalle's invariance principle* [14] by noting that the set $\mathcal{P}(\mathcal{V})$ is invariant for the system (1). □

The above lemma implies that if negative transition rates are admissible, then there exists a linear feedback law, $\{k_e\}_{e \in \mathcal{E}}$, such that $k_e(\mathbf{x}^d) = 0$ for each $e \in \mathcal{E}$ and the desired equilibrium point is locally exponentially stable.

A desirable property of the control system (1) is that stabilization of the target equilibrium can be achieved using a linear feedback law that satisfies positivity constraints away from equilibrium and is zero at equilibrium. However, any stabilizing linear control law that is zero at equilibrium must in fact be zero everywhere, if non-negativity constraints are imposed on the control inputs. On the other hand, in the next theorem we show that whenever $\mathcal{G}$ is bidirected, any feedback control law that violates positivity constraints can be implemented using a rational feedback law of the form $k(\mathbf{x}) = a(\mathbf{x}) + b(\mathbf{x})\frac{f(\mathbf{x})}{g(\mathbf{x})}$, such that $k(\mathbf{x})$ satisfies the positivity constraints and is zero at equilibrium. Note, however, that we cannot achieve the same result as in the following theorem just by setting $u_{ij}(t) = u_{ji}(t)$ whenever $u_{ij}(t)$ is negative, as the resulting solution of the system for the modified control input might not be the same.

Lemma 5.4 *Let* $\mathcal{G}$ *be a bidirected graph. Let* $k_e : \mathbb{R}^M \to (-\infty, \infty)$ *be a map for each* $e \in \mathcal{E}$ *such that there exists a unique global solution of the system* (9)*. Additionally, assume that* $\mathbf{x}(t) \in \text{int}(\mathcal{P}(\mathcal{V}))$ *for each* $t \in [0, \infty)$*. Consider the functions* $m_e^p : \mathbb{R}^M \to \{0, 1\}$ *and* $m_e^n : \mathbb{R}^M \to \{0, 1\}$*, defined as follows for each* $e \in \mathcal{E}$*:*

$$\begin{aligned} m_e^p(\mathbf{y}) &= 1 \quad \text{if} \quad k_e(\mathbf{y}) \geq 0, \quad 0 \ \text{otherwise;} \\ m_e^n(\mathbf{y}) &= 1 \quad \text{if} \quad k_e(\mathbf{y}) \leq 0, \quad 0 \ \text{otherwise.} \end{aligned}$$

Let $c_e : \mathbb{R}^M \to [0, \infty)$ *be given by*

$$c_e(\mathbf{y}) = m_e^p(\mathbf{y})k_e(\mathbf{y}) - m_{\tilde{e}}^n(\mathbf{y})k_{\tilde{e}}(\mathbf{y})\frac{y_{T(e)}}{y_{S(e)}}. \tag{12}$$

Then the solution $\tilde{\mathbf{x}}(t)$ *of the following system,*

$$\dot{\tilde{\mathbf{x}}} = \sum_{e \in \mathcal{E}} c_e(\tilde{\mathbf{x}}(t))\mathbf{B}_e\tilde{\mathbf{x}}(t), \quad \tilde{\mathbf{x}}(0) = \mathbf{x}^0 \in \text{int}(\mathcal{P}(\mathcal{V})), \tag{13}$$

is unique, defined globally, and satisfies $\tilde{\mathbf{x}}(t) = \mathbf{x}(t)$ *for all* $t \in [0, \infty)$*.*

Proof. This follows by noting that the right-hand sides of systems (9) and (13) are equal for all $t \geq 0$. □

In the above theorem, it is required that $\tilde{\mathbf{x}}(t) \in \mathrm{int}(\mathcal{P}(\mathcal{V}))$ for all $t \in [0, \infty)$. This assumption on the initial distribution $\mathbf{x}^0$ can be avoided if one uses polynomial feedback instead, as shown in the following theorem.

Proposition 5.5 *Let $\mathcal{G}$ be a bidirected graph. Suppose that $\mathbf{x}^d \in \mathrm{int}(\mathcal{P}(\mathcal{V}))$. Let $k_e : \mathbb{R}^M \to [0, \infty)$ be given by*

$$k_e(\mathbf{y}) = [(y_{S(e)} - x^d_{S(e)})^2 + (y_{T(e)} - x^d_{T(e)})^2]/x^d_{S(e)} \quad (14)$$

in system (9), for each $e \in \mathcal{E}$ and each $\mathbf{y} \in \mathbb{R}^M$. Then $\mathbf{x}^d$ is the globally asymptotically stable equilibrium point of system (9).

Before we present the proof, to facilitate our analysis, we rewrite system (9) as

$$\dot{\mathbf{x}}(t) = \mathbf{G}(\mathbf{x}(t))\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}^0 \in \mathcal{P}(\mathcal{V}) \quad (15)$$

where $\mathbf{G} : \mathbb{R}^M \to \mathbb{R}^{M \times M}$ is given by $\mathbf{G}(\mathbf{y}) = \sum_{e \in \mathcal{E}} k_e(\mathbf{y})\mathbf{B}_e$ for all $\mathbf{y} \in \mathbb{R}^M$. It is clear that when $x_{S(e)} = x^d_{S(e)}$ and $x_{T(e)} = x^d_{T(e)}$ for all $e \in \mathcal{E}$, $\mathbf{G}(\mathbf{x}^d) = \mathbf{0}$, which satisfies our requirement that the control inputs equal zero at equilibrium.

Proof. To prove the stability of system (15), we will again invoke LaSalle's invariance principle [14]. Consider the continuously differentiable function $V : \mathbb{R}^M \to \mathbb{R}_{\geq 0}$ defined in equation (11). To apply LaSalle's invariance principle, $\dot{V}(\mathbf{x})$ along the solutions $\mathbf{x}(t)$ of system (15) is required to be negative. We can compute this derivative as:

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{D}(\mathbf{x} - \mathbf{x}^d) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^d)^T\mathbf{D}\dot{\mathbf{x}} \\ &= \frac{1}{2}(\mathbf{x}^T\mathbf{G}(\mathbf{x})\mathbf{D}\mathbf{x} + \mathbf{x}^T\mathbf{D}\mathbf{G}(\mathbf{x})^T\mathbf{x} \\ &\quad - \mathbf{x}^T\mathbf{G}(\mathbf{x})\mathbf{D}\mathbf{x}^d - (\mathbf{x}^d)^T\mathbf{D}\mathbf{G}(\mathbf{x})^T\mathbf{x}).\end{aligned}$$

A simple computation shows that the last two terms in the expression above are zero. The sum of the first two terms is strictly negative; this can be confirmed by algebraic manipulation of the sum as follows. Setting $\mathbf{r}(t) = [x_1(t)/x^d_1 \ldots x_M(t)/x^d_M]^T$, we obtain:

$$\begin{aligned}&\frac{1}{2}\mathbf{x}^T\mathbf{G}(\mathbf{x})\mathbf{D}\mathbf{x}(t) + \frac{1}{2}\mathbf{x}^T\mathbf{D}\mathbf{G}(\mathbf{x})^T\mathbf{x} \\ &= \frac{1}{2}\sum_{e \in \mathcal{E}} -(r_{S(e)}(t) - r_{T(e)}(t))^2\big((x_{S(e)}(t) - x^d_{S(e)})^2 + \\ &\qquad (x_{T(e)}(t) - x^d_{T(e)})^2\big). \quad (16)\end{aligned}$$

The expression (16) is a negative sum-of-squares, and thus equals zero only when $\mathbf{x}(t) = \mathbf{x}^d$. Hence, this function is strictly negative for all $\mathbf{x} \in \mathcal{P}(\mathcal{V})\backslash\{\mathbf{x}^d\}$. Moreover, the set $\mathcal{P}(\mathcal{V})$ is invariant for the closed-loop system (15) since $\mathbf{G}(\mathbf{y})$ is an essentially non-negative matrix for which each row sums to 0, for all $\mathbf{y} \in \mathcal{P}(\mathcal{V})$. It follows from LaSalle's invariance principle that $\mathbf{x}^d$ is the globally asymptotically stable equilibrium point of the closed-loop system (15) with the control inputs k_e defined in (14). □

We now extend the stabilization results in Lemma 5.3 and Proposition 5.5 to the more general case where the target distribution has a strongly connected support and is not necessarily strictly positive everywhere on $\mathcal{V}$. We will need the following preliminary results to prove these extensions.

Lemma 5.6 *Let $\mathbf{A} \in \mathbb{R}^{M \times M}$ be an essentially non-negative matrix. Let $\mathcal{S}$ be the set of elements k in $\mathcal{V}$ such that $\sum_{i \in \mathcal{V}} A^{ik} < 0$. Assume that $\mathcal{S}$ is non-empty and that $\sum_{i \in \mathcal{V}} A^{ij} \leq 0$ for all $j \in \mathcal{V}$. Additionally, suppose that for each $j \in \mathcal{V}\backslash\mathcal{S}$, there exists a sequence $(i_n)^m_{n=1} \in \mathcal{V}$ of length m such that $i_1 = j$, $i_m \in \mathcal{S}$, and $A^{i_k i_{k-1}} > 0$ for all $i_k \neq i_{k-1}$ with $k = 2, ..., m$. Then $\mathrm{spec}(\mathbf{A})$ lies in the open left half of the complex plane.*

Proof. First, we will confirm that $\mathrm{spec}(\mathbf{A})$ lies in the closed left half of the complex plane. Toward this end, let $\lambda > 0$ be large enough such that $\lambda\mathbf{I} + \mathbf{A}$ is a non-negative matrix, where $\mathbf{I}$ is the $M \times M$ identity matrix. Since each column sum of the matrix $\lambda\mathbf{I} + \mathbf{A}$ is less than or equal to λ, it follows from [20][Theorem 4.2] and [20][Theorem 1.1] that the maximal eigenvalue r of $\lambda\mathbf{I} + \mathbf{A}$ exists and is bounded from above by λ. Next, we will establish that $r \neq \lambda$. Suppose, for the sake of contradiction, that the maximal eigenvalue of $\lambda\mathbf{I} + \mathbf{A}$ is λ, and hence that $\mathbf{A}$ has an eigenvalue at 0. Then, by [20][Theorem 4.2], there exists a nonzero element of $\mathbf{v} \in \mathbb{R}^M_{\geq 0}$ such that $\mathbf{A}\mathbf{v} = \mathbf{0}$. Therefore, $\mathbf{1}^T\mathbf{A}\mathbf{v} = \sum_{i \in V}\sum_{k \in \mathcal{S}} A^{ik}v_k = 0$. Hence, since each column of $\mathbf{A}$ corresponding to $\mathcal{V}\backslash\mathcal{S}$ sums to 0, we can conclude that $(\mathbf{A}\mathbf{v})_k = 0$ for all $k \in \mathcal{S}$. Additionally, we assumed that if $j \notin \mathcal{S}$, then there exists a sequence $(i_n)^m_{n=1} \in \mathcal{V}$ of length m such that $i_1 = j$, $i_m \in \mathcal{S}$, and $A^{i_k i_{k-1}} > 0$ for all $i_k \neq i_{k-1}$ with $k = 2, ..., m$. Moreover, all the off-diagonal elements of $\mathbf{A}$ are non-negative, and $\mathbf{A}\mathbf{v} = \mathbf{0}$. Thus, it must be the case that $v_i = 0$ for all $i \in \mathcal{N}(j)$, the set of vertices that are adjacent to any vertex $j \in \mathcal{S}$. The non-negativity of the off-diagonal elements of $\mathbf{A}$ and the fact that $\mathbf{A}\mathbf{v} = \mathbf{0}$ also imply that $v_i = 0$ for all $i \in \mathcal{N}(p)$, for all $p \in \mathcal{N}(k)$ with $k \in \mathcal{S}$. Using a similar argument, we can show that since the graph $\mathcal{G}$ is strongly connected, $v_i = 0$ for all $i \in \mathcal{V}$. This implies that $r \neq \lambda$. Therefore, the matrix $\mathbf{A}$ is Hurwitz. This concludes the proof. □

Theorem 5.7 *Let $\mathbf{f} : \mathbb{R}^{M_1} \to: \mathbb{R}^{M_1}$ be a Lipschitz-continuous vector field, where M_1 is the cardinality of a set $\mathcal{V}_1 \subset \mathcal{V}$. Also, let M_2 be the cardinality of $\mathcal{V}_2 = \mathcal{V}\backslash\mathcal{V}_1$.*

Suppose there exists a continuously differentiable positive function $U : \mathbb{R}^{M_2} \to \mathbb{R}_{\geq 0}$ such that $\frac{\partial U}{\partial \mathbf{y}}\mathbf{f}(\mathbf{y}) \leq 0$, with the equalities $U(\mathbf{y}) = \frac{\partial U}{\partial \mathbf{y}}\mathbf{f}(\mathbf{y}) = 0$ holding only at a unique fixed point of $\mathbf{f}(\mathbf{x})$ given by $\mathbf{y} = \mathbf{x}^d \in \mathcal{P}(\mathcal{V}_1)$. Now consider the following system with solution $\mathbf{z}(t) \in \mathbb{R}^M$,

$$\begin{aligned}\dot{\mathbf{z}}_1(t) &= \mathbf{f}(\mathbf{z}_1(t)) + \mathbf{G}_2\mathbf{z}_2(t),\\ \dot{\mathbf{z}}_2(t) &= \mathbf{A}\mathbf{z}_2(t),\\ \mathbf{z}(0) &= \mathbf{z}^0 \in \mathcal{P}(\mathcal{V}),\end{aligned} \tag{17}$$

where $\mathbf{z}(t) = [\mathbf{z}_1(t)^T \ \mathbf{z}_2(t)^T]^T$, $\mathbf{G}_2 \in \mathbb{R}^{M \times M_2}$, $\mathbf{A} \in \mathbb{R}^{M_2 \times M_1}$, $\mathcal{V}$ has cardinality $M > M_1$, and $\mathcal{P}(\mathcal{V})$ is invariant for the system. Lastly, assume that the matrix $\mathbf{A}$ satisfies the sufficient conditions in Lemma (5.6) for spec($\mathbf{A}$) *to lie in the open left half of the complex plane. Then $\mathbf{z}^d = [(\mathbf{x}^d)^T \ \ \mathbf{0}^T]^T$ is the globally asymptotically stable equilibrium point of the system (17).*

Proof. From the proof of Lemma (5.6), the matrix $\mathbf{A}$ is Hurwitz. This implies that $\lim_{t\to\infty} \mathbf{z}_2(t) = \mathbf{0}$. Hence, $\lim_{t\to\infty} \sum_{i\in\mathcal{V}_1} (\mathbf{z}_1)_i(t) = 1$, since $\mathcal{P}(\mathcal{V})$ is invariant for the system (17). We can extend the function U to a function $\hat{U}$ on $\mathbb{R}^M$ by defining $\hat{U}(\mathbf{y}) = U(\mathbf{y}_1)$, where $\mathbf{y} = [\mathbf{y}_1^T \ \mathbf{y}_2^T]^T$, $\mathbf{y}_1 \in \mathbb{R}^{M_1}$, and $\mathbf{y}_2 \in \mathbb{R}^{M_2}$. Consider the set $\Delta_c = \{\mathbf{y} \in \mathcal{P}(\mathcal{V}) : \sum_{i\in\mathcal{V}_2} y_i \leq c\}$. From the assumptions made on U, we have that $\frac{\partial U}{\partial \mathbf{y}_1}\mathbf{f}(\mathbf{y}_1) + \frac{\partial U}{\partial \mathbf{y}_1}\mathbf{G}_2\mathbf{y}_2 \leq 0$ on the set Δ_0, with the equality holding only at $\mathbf{y} = [(\mathbf{x}^d)^T \ \ \mathbf{0}^T]^T$. Now fix $c_1 > 0$. By the continuity of the function $\hat{U}_d(\mathbf{y}) := \frac{\partial U}{\partial \mathbf{y}_1}\mathbf{f}(\mathbf{y}_1) + \frac{\partial U}{\partial \mathbf{y}_1}\mathbf{G}_2\mathbf{y}_2$, there exist $\epsilon > 0$ and $c_2 > 0$ such that $\hat{U}_d(\mathbf{y}) \leq -\epsilon$ for all $\mathbf{y} \in U^{-1}\big((c_1,\infty]\big) \cap \Delta_{c_2}$. Due to the assumption on the matrix $\mathbf{A}$ that $\sum_{i\in\mathcal{V}} A^{ij} \leq 0$ for all $j \in \mathcal{V}$, it follows that $U^{-1}\big([0,c_1]\big) \cap \Delta_{c_2}$ is invariant for the system (17). This implies that the equilibrium $\mathbf{x}^d$ is Lyapunov stable for the system (17). Next, we will establish that the distribution $\mathbf{x}^d$ is also globally attractive. We know that $\lim_{t\to\infty} \mathbf{z}_2(t) = \mathbf{0}$. Since $\mathcal{P}(\mathcal{V})$ is compact, we can conclude that there exists $t_0 \geq 0$ such that $\mathbf{z}(t) \in U^{-1}\big([0,c_1]\big) \cap \Delta_{c_2}$ for all $t \geq t_0$. The constant c_1 can be chosen to be arbitrarily small. This implies that $\lim_{\to\infty} \mathbf{z}(t) = \mathbf{x}^d$. □

Using the results in Lemma 5.6 and Theorem 5.7, we prove the following result, which generalizes Lemma 5.3 and Proposition 5.5 to target distributions that have a strongly connected support.

Theorem 5.8 *Let $\mathcal{G}$ be a bidirected graph. Suppose that $\mathbf{x}^d \in \mathcal{P}(\mathcal{V})$ has a strongly connected support. Let $\mathcal{V}_1$ be the support of $\mathbf{x}^d$ and $\mathcal{V}_2 = \mathcal{V}\backslash\mathcal{V}_1$. Let $k_e : \mathbb{R}^M \to [0,\infty)$ in system (9) be defined as*

$$k_e(\mathbf{x}) = \begin{cases} a_1(x^d_{T(e)}y_{S(e)} - x^d_{S(e)}y_{T(e)}) \\ \quad + a_2[(y_{S(e)} - x^d_{S(e)})^2 + (y_{T(e)} - x^d_{T(e)})^2]/x^d_{S(e)} \\ \quad \textit{if } S(e), T(e) \in \mathcal{V}_1, \\ g_e \in (0,\infty) \quad \textit{if } S(e) \in \mathcal{V}_2, \\ 0 \quad \textit{if } S(e) \in \mathcal{V}_1,\ T(e) \in \mathcal{V}_2, \end{cases}$$

where either $a_1 \neq 0$ or $a_2 \neq 0$; that is, only one of the control laws (10) or (14) is used to stabilize the system. Then $\mathbf{x}^d$ is the globally asymptotically stable equilibrium point of the system (9).

Proof. Without loss of generality, we can assume that $\mathcal{V}_1$ is of the form $\{1, ..., M_1\}$ for some $M_1 \leq M$. For this analysis we use the modified system (15). Since $k_e(\mathbf{y}) = 0$ whenever $S(e) \in \mathcal{V}_1,\ T(e) \in \mathcal{V}_2$, the state-dependent matrix $\mathbf{G}^T$ can be factorized into the form

$$\mathbf{G}(\mathbf{y}) = \begin{bmatrix} \mathbf{G}_1(\mathbf{y}_1) & \mathbf{G}_2 \\ \mathbf{0} & \mathbf{A} \end{bmatrix}, \tag{18}$$

where $\mathbf{G}_1 : \mathbb{R}^{M_1} \to \mathbb{R}^{M_1 \times M_1}$ and $\mathbf{G}_2 \in \mathbb{R}^{M_1 \times M_2}$. Moreover, since the graph $\mathcal{G}$ is strongly connected and bidirected, from the definition of k_e, it follows that $\mathbf{A}$ satisfies the sufficient conditions of Lemma 5.6; therefore, spec($\mathbf{A}$) lies in the open left half of the complex plane. In addition, since each column of the matrix $\mathbf{G}(\mathbf{y})$ sums to 0 and this matrix is essentially non-negative for each $\mathbf{y} \in \mathcal{P}(\mathcal{V})$, the set $\mathcal{P}(\mathcal{V})$ is invariant for the system (15). Let M_1 be the cardinality of the set $\mathcal{V}_1$. Additionally, define the function $U : \mathbb{R}^{M_1} \to \mathbb{R}_{\geq 0}$ by $U(\mathbf{y}) = \frac{1}{2}(\mathbf{y} - \mathbf{y}^d)^T\mathbf{D}(\mathbf{y} - \mathbf{y}^d)$ for all $\mathbf{y} \in \mathbb{R}^{M_1}$, where $\mathbf{y}^d \in \mathrm{int}(\mathcal{P}(\mathcal{V}_1))$ such that $\mathbf{x}^d = [(\mathbf{y}^d)^T \ \mathbf{0}^T]^T \in \mathcal{P}(\mathcal{V})$, and $\mathbf{D} \in \mathbb{R}^{M_1 \times M_1}$ is given by $\mathbf{D} = [\mathrm{diag}(\mathbf{x}^d)]^{-1}$. By Lemma 5.3 and Proposition 5.5, this function satisfies the conditions of Theorem 5.7 with respect to the vector field $\mathbf{f}(\mathbf{z}) = \mathbf{G}_1^T(\mathbf{z})\mathbf{z}$ on the set $\mathcal{P}(\mathcal{V}_1)$. Then the result follows from Theorem 5.7. □

6 CONCLUSION AND FUTURE WORK

We proved a number of controllability and stabilization results for Kolmogorov forwards equations of continuous-time Markov chains. Since an implementation of these control laws on robots requires discretization of time, a potential direction of future work would be to extend these results to the discrete-time case considered in [1].

References

[1] Behcet Acikmese and David S. Bayard. A Markov chain approach to probabilistic swarm guidance. In *American Control Conference (ACC)*, pages 6300–6307. IEEE, 2012.

[2] Spring Berman, Ádám Halász, M. Ani Hsieh, and Vijay Kumar. Optimized stochastic policies for task allocation in swarms of robots. *IEEE Transactions on Robotics*, 25(4):927–937, 2009.

[3] Anthony Bloch, John Baillieul, Peter Crouch, Jerrold E. Marsden, Dmitry Zenkov, P.S. Krishnaprasad, and Richard M. Murray. *Nonholonomic mechanics and control*, volume 24. Springer, 2003.

[4] William M Boothby and Edward N Wilson. Determination of the transitivity of bilinear systems. *SIAM journal on control and optimization*, 17(2):212–221, 1979.

[5] Airlie Chapman. Advection on graphs. In *Semi-Autonomous Networks*, pages 3–16. Springer, 2015.

[6] Francis H. Clarke, Yuri S. Ledyaev, Eduardo D. Sontag, and Andrei I. Subbotin. Asymptotic controllability implies feedback stabilization. *IEEE Transactions on Automatic Control*, 42(10):1394–1407, 1997.

[7] Vaibhav Deshmukh, Karthik Elamvazhuthi, Shiba Biswal, Zahi Kakish, and Spring Berman. Mean-field stabilization of Markov chain models for robotic swarms: Computational approaches and experimental results. *IEEE Robotics and Automation Letters*, 3(3):1985–1992, 2018.

[8] Karthik Elamvazhuthi and Spring Berman. Mean-field models in swarm robotics: a survey. *Bioinspiration & Biomimetics*, 15(1):015001, 2019.

[9] Karthik Elamvazhuthi, Matthias Kawski, Shiba Biswal, Vaibhav Deshmukh, and Spring Berman. Mean-field controllability and decentralized stabilization of Markov chains. In *IEEE Conference on Decision and Control (CDC)*. IEEE, 2017.

[10] Karthik Elamvazhuthi, Hendrik Kuiper, Matthias Kawski, and Spring Berman. Bilinear controllability of a class of advection-diffusion-reaction systems. *IEEE Transactions on Automatic Control*, 2018. doi: 10.1109/TAC.2018.2885231.

[11] David Elliott. *Bilinear control systems: matrices in action*, volume 169. Springer Science & Business Media, 2009.

[12] Heiko Hamann and Heinz Wörn. A framework of space–time continuous models for algorithm design in swarm robotics. *Swarm Intelligence*, 2(2-4):209–239, 2008.

[13] M Ani Hsieh, Ádám Halász, Spring Berman, and Vijay Kumar. Biologically inspired redistribution of a swarm of robots among multiple sites. *Swarm Intelligence*, 2(2):121–141, 2008.

[14] Hassan K. Khalil. *Nonlinear systems*. Pearson, 3rd edition, 2001.

[15] Jerzy Klamka. Constrained controllability of nonlinear systems. *Journal of Mathematical Analysis and Applications*, 201(2):365–374, 1996.

[16] Kristina Lerman, Chris Jones, Aram Galstyan, and Maja J. Matarić. Analysis of dynamic task allocation in multi-robot systems. *International Journal of Robotics Research*, 25(3):225–241, 2006.

[17] Jr-Shin Li, Wei Zhang, and Liang Wang. Computing controllability of systems on SO(n) over graphs. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 5511–5516. IEEE, 2017.

[18] T. William Mather and M. Ani Hsieh. Synthesis and analysis of distributed ensemble control strategies for allocation to multiple tasks. *Robotica*, 32(02):177–192, 2014.

[19] Mehran Mesbahi and Magnus Egerstedt. *Graph theoretic methods in multiagent networks*. Princeton University Press, 2010.

[20] Henryk Minc. *Nonnegative matrices*. John Wiley & Sons, New York, 1988.

[21] Héctor J Sussmann. Subanalytic sets and feedback control. *J. Differential Equations*, 31(1):31–52, 1979.