# Adaptation of Gradient-based Navigation Control for Holonomic Robots to Nonholonomic Robots 

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#### Abstract

In this paper, we propose a gradient-based nonlinear control approach for stabilizing a nonholonomic Wheeled Mobile Robot (WMR) to a target position in environments with and without obstacles. This approach enables any gradient-based feedback control law (with bounded or unbounded gradients) developed for a holonomic point-mass robot model to be adapted to control a nonholonomic robot. The proposed controller is defined in terms of smooth continuous functions, which produce smooth robot trajectories and can be tuned to stabilize the robot to the goal position at a desired convergence rate. We first prove that the controller will stabilize a nonholonomic robot to a target point in an obstacle-free environment. To stabilize the robot's position in environments with obstacles, we modify our controller to utilize the gradient of an artificial potential function and use Lyapunov stability theory to prove that the robot is guaranteed to converge to the target position under this controller. We demonstrate the effectiveness of our controller for various initial robot positions and environments, and two types of potential fields that are widely used in gradient-based methods for obstacle avoidance, through MATLAB simulations and experiments with a commercial nonholonomic WMR.


Index Terms-Wheeled Mobile Robots, Nonholonomic Constraint, Nonlinear Controller, Obstacle Avoidance, Artificial Potential Field.

## I. Introduction

MOTION control of nonholonomic Wheeled Mobile Robots (WMRs) is required in a wide variety of applications in robotics. Despite extensive work on designing motion controllers for nonholonomic WMRs, challenges still arise due to the nonholonomic constraints in the robot's kinematic model [1], [2]. Existing control approaches for stabilizing the position of a nonholonomic robot suffer from various limitations, including (1) chattering in the robot's motion that results from the use of discontinuous functions in the control law, e.g. sgn [3], arctan [4], and atan2; and (2) erratic or oscillatory transient robot motions, which are intrinsic characteristics of time-varying control laws [5] and pure geometric techniques [6]. Furthermore, many existing control approaches for obstacle avoidance (1) have been developed using a holonomic motion planner, which may introduce infeasible collision-free paths and cannot be implemented

[^0]on nonholonomic robots [7]; (2) do not have mathematical guarantees on performance; and (3) can result in the robot becoming trapped in a local minimum.

There are also many gradient-based control approaches in the literature that are rigorously proved to achieve position stabilization and obstacle avoidance for holonomic WMRs, but cannot be directly applied to nonholonomic WMRs. This challenge motivated our work in this paper, which makes the following main contributions. (1) We present an approach to converting any gradient-based feedback controller designed for holonomic robots into a controller that can be implemented on nonholonomic robots for position stabilization and obstacle avoidance. (2) Our controller has no discontinuities, since it utilizes continuous functions (namely, trigonometric functions of the robot's heading angle) that produce a smooth robot trajectory. (3) Using Lyapunov stability theory, we prove that the controller is asymptotically stable for sets of both positive and negative controller gains, given any initial robot configuration. Thus, the robot can drive either forward or backward to the target position as needed.

## II. Related Work

Various controllers have been designed to stabilize a nonholonomic WMR to a target position. Discontinuous controllers were developed in [3], [4] for exponential position stabilization. Since these controllers are based on a switching framework, noise in the robot's sensor measurements can result in undesired chattering along the boundaries of the switching conditions [8]. A time-varying feedback stabilization controller was proposed in [9], although it can produce transient oscillations and undesired cusps in the trajectory of the robot as it converges to the target position [8]. Control approaches based on Lyapunov stability theory were developed in [10], [11]; however, these approaches cannot produce smooth continuous time-invariant feedback laws [8].
A feedback linearization method was proposed in [12] for smooth position stabilization, but this method produces a singularity in the control law when the robot's linear velocity is zero. Nonlinear Model Predictive Control (NMPC) methods for position stabilization have also been investigated [5], [13]. However, online implementations of these methods are computationally intensive, and the transient portion of the robot's trajectory will not be smooth if the prediction horizon is not large enough. Geometric control techniques developed for holonomic systems, e.g. the approaches in [14], [15], cannot be applied to nonholonomic robots, given their no-slip velocity constraint. To address this issue, a geometric control approach for position stabilization of noholonomic WMRs was designed in [16] by exploiting properties of exponential coordinates and
the special Euclidean group $S E(2)$. However, in implementation, the controller's dependence on direct measurements of the robot's heading angle can produce chattering when this angle is near $\pi$ or $-\pi$ rad. Moreover, discontinuities arise in the control law due to the use of the atan 2 function.

Obstacle avoidance is another well-studied challenging problem in the control of nonholonomic WMRs. Many existing controllers for obstacle avoidance by WMRs are based on artificial potential fields [17], [18]. In these approaches, the potential field is constructed to produce virtual attraction forces on the robot toward the target position and virtual repulsion forces away from the obstacles [19]. These methods do not ensure global convergence to the target point while avoiding obstacles, since the robot can become trapped in local minima [20]. When applied to a nonholonomic WMR, potential field-based controllers that have been developed for holonomic robots may result in infeasible robot trajectories, since the robot cannot necessarily move along the prescribed gradient due to its nonholonomic constraint. In [21], [22], [23], methods are proposed for mapping the infeasible trajectories obtained by the holonomic planner to feasible ones for nonholonomic robots; however, the potential functions in these methods are not strictly decreasing. To overcome this drawback, approaches that use differential flatness [24] and iterative calculations [21] have been proposed to derive feasible collision-free trajectories for nonholonomic WMRs.

A feedback controller for obstacle avoidance by nonholonomic WMRs was designed in [25] using a time-varying potential function with no local minima or saddle points. This approach is subject to the aforementioned issues associated with discontinuous time-varying controllers. A special type of potential function called a navigation function [18] guarantees convergence of a robot to a target position, defined as the global minimum of the function, and ensures no collisions with obstacles. The design of navigation function-based controllers for nonholonomic WMRs was studied in [26], [27], [28]. These methods rely on prior knowledge about the environment, including the geometry of the obstacles and the domain. In addition to potential forces, gyroscopic forces were proposed in [29] to implement obstacle avoidance while ensuring that the robot converges to the target position. In [30], a reactive control method inspired by magnetic fields was proposed for obstacle avoidance by nonholonomic WMRs in environments with convex obstacles. This method and the sliding mode controller in [31] guarantee almost global convergence to the target position, can be applied in unbounded convex domains, and require the robot to have only local sensing capabilities, with no prior information about the environment such as the locations and shapes of the obstacles.

## III. Problem Statement

We consider a nonholonomic WMR with a reference point $P$ at the midpoint of the axis connecting its wheels, as shown in Fig. 1. The robot's configuration at time $t$ is defined as $\boldsymbol{\xi}(t)=\left[\boldsymbol{x}(t)^{T} \theta(t)\right]^{T}=[x(t) y(t) \theta(t)]^{T} \in \mathbb{R}^{3}$, where $x$ and $y$ denote the coordinates of point $P$ in the global coordinate frame and $\theta$ denotes the robot's heading angle, defined as the


Fig. 1: (a) 3-D view of the Turtlebot3 Burger robot [32], a nonholonomic WMR that we used in our experiments, and (b) definitions of the state variables $\rho, \alpha$, and $\theta$ in an overhead view of the robot.
angle of the robot's heading direction with respect to the $x_{o^{-}}$ axis of the global frame. The unicycle kinematic model of the robot is given by:

$$
\begin{equation*}
\dot{x}=v \cos (\theta), \quad \dot{y}=v \sin (\theta), \quad \dot{\theta}=\omega \tag{1}
\end{equation*}
$$

where $v$ is the speed of the reference point and $\omega$ is the angular velocity of the robot. The vector of control inputs is defined as $\boldsymbol{u}=[v \omega]^{T} \in \mathbb{R}^{2}$. We assume that the robot knows its initial configuration $\boldsymbol{\xi}(0)$ and can use its sensors to determine $\boldsymbol{x}(t)$ and the quaternion representation [33] of its heading angle $\theta(t)$ at each time $t>0$. A quaternion $\boldsymbol{q}$ is defined as the sum of a scalar $q_{w}$ and a vector $\left[\begin{array}{lll}q_{x} & q_{y} & q_{z}\end{array}\right]^{\mathrm{T}}$ described in an orthonormal basis $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ of $\mathbb{R}^{3}$ :

$$
\begin{equation*}
\boldsymbol{q}=q_{w}+q_{x} \boldsymbol{i}+q_{y} \boldsymbol{j}+q_{z} \boldsymbol{k} \tag{2}
\end{equation*}
$$

Since the WMR moves in the $x-y$ plane, the elements of $\boldsymbol{q}$ are given by:

$$
\begin{equation*}
q_{x}=0, \quad q_{y}=0, \quad q_{z}=\sin \left(\frac{\theta}{2}\right), \quad q_{w}=\cos \left(\frac{\theta}{2}\right) \tag{3}
\end{equation*}
$$

two of which are trigonometric functions of the robot's heading. We will design controllers for the robot that solve the following problems. In both problems, we define the target position of the robot (specifically, of the point $P$ ) as the origin of the global coordinate frame, without loss of generality.

Problem III.1. Consider a robot with model (1) that moves in an unbounded domain. Design a feedback control law of the form $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{\xi})$ that requires only trigonometric functions of $\theta$ and drives the robot to the target position from any initial configuration.
Problem III.2. Consider a robot with model (1) that moves in a convex domain that contains one or more convex obstacles. Design a controller of the form $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{\xi})$, based on the gradient of a potential field, that requires only trigonometric functions of $\theta$ and drives the robot to the target position from
almost ${ }^{1}$ any initial configuration while preventing collisions with the obstacles and the domain's boundary.

Existing solutions to the above problems have the limitations described in Section I. In particular, they require the controller to be an explicit function of the Euler angle representation of the robot's heading, which can result in large changes in the control input when the robot's heading increases past $\pi \mathrm{rad}$ and is measured as $\theta \in[-\pi, 0] \mathrm{rad}$, or when it decreases past $-\pi \mathrm{rad}$ and is measured as $\theta \in[0, \pi] \mathrm{rad}$. These large changes can cause undesired transient behaviors in the robot's navigation, such as sudden reorientations of the robot and chattering in its trajectory that is characteristic of switching controllers [34]. In order to produce a smooth robot trajectory, our position controller uses trigonometric functions of the robot's heading, i.e. $\sin (\theta)$ and $\cos (\theta)$, in accordance with the quaternion representation Eq. (3). Moreover, our approach to collision-free position control can implement any gradient-based controller designed for a holonomic robot on a nonholonomic WMR. We note that our gradient-based controller inherits (and cannot alter) the properties and limitations of the corresponding controller for holonomic robots, such as dependence on prior information about the environment or the existence of local minima that can trap the robot.

## IV. Control System

## A. Controller Design

Defining $\rho:=\sqrt{x^{2}+y^{2}}$ and $\alpha:=\tan ^{-1}\left(\frac{y}{x}\right)$, we can represent the position of the reference point $P$ in a polar coordinate system as $x=\rho \cos (\alpha)$ and $y=\rho \sin (\alpha)$. Accordingly, we can rewrite the model in Eq. (1) in the polar coordinate system as:

$$
\begin{align*}
& \dot{\rho} \cos (\alpha)-\rho \dot{\alpha} \sin (\alpha)=v \cos (\theta), \\
& \dot{\rho} \sin (\alpha)+\rho \dot{\alpha} \cos (\alpha)=v \sin (\theta), \\
& \dot{\theta}=\omega \tag{4}
\end{align*}
$$

We multiply the first and second equations in Eq. (4) by $\cos (\alpha)$ and $\sin (\alpha)$, respectively, and add them up. Then we repeat this procedure by multiplying the first and second equations by $-\sin (\alpha)$ and $\cos (\alpha)$, respectively, and adding them up. As a result, we obtain the following representation of the unicycle kinematic model in the polar coordinate system, where $\rho, \alpha$, and $\theta$ are state variables and $v$ and $\omega$ are control inputs:

$$
\begin{equation*}
\dot{\rho}=v \cos (\theta-\alpha), \quad \dot{\alpha}=\frac{v}{\rho} \sin (\theta-\alpha), \quad \dot{\theta}=\omega \tag{5}
\end{equation*}
$$

We propose the following control laws for $v$ and $\omega$ :

$$
\begin{equation*}
v=k_{v} \rho, \quad \omega=k_{\omega} \sin (\theta-\alpha) \tag{6}
\end{equation*}
$$

where $k_{v}$ and $k_{\omega}$ are controller gains. Given Eq. (3) and the definition of $\rho$, Eq. (6) can be rewritten as:

$$
\begin{align*}
v & =k_{v} \sqrt{x^{2}+y^{2}}=k_{v}\|\boldsymbol{x}\| \\
\omega & =k_{\omega} \sin (\theta-\alpha)  \tag{7}\\
& =k_{\omega} \sin (\theta) \cos (\alpha)-k_{\omega} \cos (\theta) \sin (\alpha) \\
& =2 k_{\omega} q_{z} q_{w} \cos (\alpha)-k_{\omega}\left(q_{w}^{2}-q_{z}^{2}\right) \sin (\alpha) \tag{8}
\end{align*}
$$

[^1]

Fig. 2: The phase portrait of the closed-loop system when the (a) negative and (b) positive controller gains are used.

Remark IV.1. As Eq. (7) shows, we can directly substitute the position and orientation measurements from the robot's sensors, i.e. $x, y, q_{z}$, and $q_{w}$, into the control laws in order to calculate the control inputs without any post-processing of these measurements. Moreover, our proposed control law for $\omega$ does not have any discontinuities in the robot's heading angle $\theta$, since it is based on the quaternion representation of rotations using trigonometric functions of the heading angle measurements.

## B. Convergence Analysis

To analyze the stability of the proposed control system, we substitute the control law (6) into the robot's kinematic model (5) and obtain the equations of the closed-loop system as:

$$
\begin{align*}
& \dot{\rho}=k_{v} \rho \cos (\theta-\alpha) \\
& \dot{\alpha}=k_{v} \sin (\theta-\alpha) \\
& \dot{\theta}=k_{\omega} \sin (\theta-\alpha) \tag{9}
\end{align*}
$$

Furthermore, defining $\beta:=\theta-\alpha$ and subtracting the second equation from the third equation in (9), we obtain a reduced-
dimensional representation of the closed-loop system:

$$
\begin{align*}
& \dot{\rho}=k_{v} \rho \cos (\beta)  \tag{10}\\
& \dot{\beta}=\left(k_{\omega}-k_{v}\right) \sin (\beta) \tag{11}
\end{align*}
$$

The equilibrium points of this system are located where $\rho=0$ and $\beta=n \pi$ for $n \in \mathbb{Z}$. By linearizing the system about these points, we can confirm that if $k_{\omega}<k_{v}<0$, then the equilibrium points with even values of $n$ are locally exponentially stable, while the equilibria with odd values of $n$ are unstable. A converse statement about even and odd values of $n$ holds true if $0<k_{v}<k_{\omega}$. The next theorem characterizes the region of attraction of the stable equilibrium points.

Theorem IV.2. Consider the closed-loop system (10)-(11). If $k_{\omega}<k_{v}<0$, then an equilibrium point of this system, which is specified by an even number for $n$ as $n=2 m, m \in \mathbb{Z}$, is asymptotically stable, and the set $\mathcal{R}=\{\boldsymbol{\xi} \mid \rho>0,(2 m-1) \pi<\beta<(2 m+1) \pi\}$ is its basin of attraction.
Proof. Using the trigonometric identity $\sin (\beta)=\frac{2 \tan \left(\frac{\beta}{2}\right)}{1+\tan ^{2}\left(\frac{\beta}{2}\right)}$ and the following change of variable,

$$
\begin{equation*}
\eta=\tan \left(\frac{\beta}{2}\right) \rightarrow \dot{\eta}=\frac{1}{2} \dot{\beta}\left(1+\tan ^{2}\left(\frac{\beta}{2}\right)\right) \tag{12}
\end{equation*}
$$

Eq. (11) is simplified to

$$
\begin{equation*}
\dot{\eta}=k_{\beta} \eta \tag{13}
\end{equation*}
$$

where $k_{\beta}:=k_{\omega}-k_{v}$. The solution of Eq. (13) is

$$
\begin{equation*}
\eta(t)=\eta_{0} e^{k_{\beta} t} \tag{14}
\end{equation*}
$$

where $\eta_{0}$ denotes the value of $\eta$ at $t=0$. This shows that if $k_{\beta}<0, \eta(t)$ exponentially converges to zero for any value of $\eta_{0}$. This assures monotonic convergence of $\beta$ to $2 m \pi$ from any value in $((2 m-1) \pi,(2 m+1) \pi)$, since $\tan (\cdot)$ is a strictly monotonic function on $\left(\frac{(2 m-1) \pi}{2}, \frac{(2 m+1) \pi}{2}\right)$. In Fig. 2a, the projection of the system trajectories for $m=-1,0,1$ along the vertical axis is monotonic, which shows this monotonic evolution of $\beta$. Furthermore, using the trigonometric identity $\cos (\beta)=\frac{1-\tan ^{2}\left(\frac{\beta}{2}\right)}{1+\tan ^{2}\left(\frac{\beta}{2}\right)}$ and Eq. (14), we can rewrite Eq. (10) as

$$
\begin{equation*}
\dot{\rho}=f(t) \rho, \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
f(t)=k_{v}\left(\frac{1-\eta_{0}^{2} e^{2 k_{\beta} t}}{1+\eta_{0}^{2} e^{2 k_{\beta} t}}\right) \tag{16}
\end{equation*}
$$

Eq. (15) is a first-order linear differential equation with a timevarying coefficient, whose solution is given by

$$
\begin{equation*}
\rho(t)=\rho_{0} e^{\int f(t) d t} \tag{17}
\end{equation*}
$$

where $\rho_{0}$ denotes the value of $\rho$ at $t=0$. To solve the integral in the exponential term, we use the following change of variable:

$$
\begin{equation*}
\zeta(t):=\eta_{0}^{2} e^{2 k_{\beta} t} \rightarrow d \zeta=2 k_{\beta} \eta_{0}^{2} e^{2 k_{\beta} t} d t \tag{18}
\end{equation*}
$$

Then we have that $\int f(t) d t$ is equal to:

$$
\begin{align*}
\int k_{v} \frac{1-\zeta(t)}{1+\zeta(t)} d t & =\frac{k_{v}}{2 k_{\beta}} \int \frac{1-\zeta}{1+\zeta} \frac{d \zeta}{\zeta} \\
& =\frac{k_{v}}{2 k_{\beta}} \int\left(\frac{1}{\zeta}-\frac{2}{1+\zeta}\right) d \zeta \\
& =\frac{k_{v}}{2 k_{\beta}}(\ln \zeta-2 \ln (1+\zeta)) \\
& =\ln \left(\frac{\zeta}{(1+\zeta)^{2}}\right)^{\frac{k_{v}}{2 k_{\beta}}} \tag{19}
\end{align*}
$$

Substituting the expression for $\int f(t) d t$ from Eq. (19) into Eq. (17), we obtain the solution for $\rho(t)$ as

$$
\begin{equation*}
\rho(t)=\rho_{0}\left(\frac{\eta_{0}^{2} e^{2 k_{\beta} t}}{\left(1+\eta_{0}^{2} e^{2 k_{\beta} t}\right)^{2}}\right)^{\frac{k_{v}}{2 k_{\beta}}} \tag{20}
\end{equation*}
$$

If $k_{\beta}<0$, we can confirm that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\eta_{0}^{2} e^{2 k_{\beta} t}}{\left(1+\eta_{0}^{2} e^{2 k_{\beta} t}\right)^{2}}=0 \tag{21}
\end{equation*}
$$

which means that $\rho(t) \rightarrow 0$ as $t \rightarrow \infty$ if $\frac{k_{v}}{k_{\beta}}>0$. Therefore, $\rho(t)$ converges to zero for any value of $\rho_{0}$ if $k_{v}, k_{\beta}<0$, which is equivalent to $k_{\omega}<k_{v}<0$. This completes the proof.

Corollary IV.3. The equilibrium points of the closed-loop system are marked by the green circles in Figs. $2 a$ and $2 b$. If we set $0<k_{v}<k_{\omega}$, then the equilibrium points that are specified by odd values of $n$ become asymptotically stable, and the others become unstable. This occurs if $\beta$ is replaced by $\beta+\pi$ in Eqs. (10)-(11) and the procedure in the proof of Theorem IV. 2 is reapplied. The evolution of the system's trajectories for positive values of $k_{v}$ and $k_{\omega}$ is illustrated in Fig. 2b. Monotonic convergence of $\beta$ to $(2 m+1) \pi$ for $m=-1,0,1$, and convergence of $\rho$ to zero, are seen along the vertical and horizontal axes, respectively.

Remark IV.4. As shown in the phase portraits in Fig. $2 a$ and Fig. 2b, the robot's radial coordinate $\rho$ diverges to infinity if and only if the initial angle $\beta(0)$ is exactly $(2 m+1) \pi \mathrm{rad}$ when using negative controller gains or $2 m \pi$ rad when using positive controller gains (for $m \in \mathbb{Z}$ ). To ensure that the robot converges to the target point from such initial configurations, positive controller gains can be used when $\beta(0)=(2 m+1) \pi$ (cases 7 and 10 in Fig. 3), and negative gains can be used when $\beta(0)=2 m \pi$ (cases 4 and 6 in Fig. 3).

Theorem IV.5. Consider a nonholonomic WMR with kinematic model (1) that moves in a convex domain containing one or more convex obstacles. Suppose there exists an artificial potential field $\varphi=\varphi(\boldsymbol{x})$ that admits a global minimum at the origin $\boldsymbol{x}=\mathbf{0}$ and for which $\mu:=\|\nabla \varphi\|$ is maximal on the boundaries of the obstacles. If $k_{v}<0$ and $k_{\omega}=2 k_{v}$, then the following control law drives the robot to the origin from almost any initial position in the domain while preventing it from colliding with the obstacles and the domain's boundary:

$$
\begin{equation*}
v=k_{v} \mu, \quad \omega=k_{\omega} \frac{\mu}{\rho} \sin (\theta-\alpha) \tag{22}
\end{equation*}
$$

Proof. Inserting the control law in Eq. (22) into the robot's kinematic model in Eq. (5), and using the fact that $\beta=\theta-\alpha$, the equations of the closed-loop system become:

$$
\begin{equation*}
\dot{\rho}=k_{v} \mu \cos (\beta), \quad \rho \dot{\beta}=k_{v} \mu \sin (\beta) \tag{23}
\end{equation*}
$$

We define a set $\mathcal{C}$ as

$$
\begin{equation*}
\mathcal{C}=\left\{(\rho, \beta) \in \mathbb{R}^{2} \mid \sin (\beta)=0 \quad \vee \cos (\beta)=0\right\} \tag{24}
\end{equation*}
$$

If the system trajectories start from this set, then the solution of Eq. (23) is straightforward to calculate. We will now consider the time evolution of trajectories that start outside this set.

We note that $\mu$ is implicitly a function of $\rho$, and the equilibrium points of system (23) are the elements of the set

$$
\begin{equation*}
\mathcal{E}:=\left\{(\rho, \beta) \in \mathbb{R}^{2} \mid \mu=0\right\} . \tag{25}
\end{equation*}
$$

This set also represents the critical point of the potential field $\varphi(\boldsymbol{x})$. We now consider the function

$$
\begin{equation*}
V=\rho \tan ^{2}(\beta / 2) \tag{26}
\end{equation*}
$$

which equals zero if $\rho=0$ and/or $\beta=2 m \pi, m \in \mathbb{Z}$, and is positive otherwise. The time derivative of $V$ along the trajectories of the closed-loop system is

$$
\begin{equation*}
\dot{V}=k_{v} \mu(2+\cos (\beta)) \tan ^{2}(\beta / 2) \tag{27}
\end{equation*}
$$

which is zero when $\mu=0$ and/or $\beta=2 m \pi, m \in \mathbb{Z}$, and is negative otherwise. Invoking LaSalle's invariance principle [35], we can conclude that system trajectories that start outside the set $\mathcal{C}$ converge to the largest invariant set in the set

$$
\begin{equation*}
\mathcal{S}:=\left\{(\rho, \beta) \in \mathbb{R}^{2} \mid \mu=0 \vee \beta=2 m \pi, m \in \mathbb{Z}\right\} \tag{28}
\end{equation*}
$$

which contains $\mathcal{E}$. Points $(\rho, \beta)$ for which $\rho>0$ and $\beta=2 m \pi$, $m \in \mathbb{Z}$, do not comprise an invariant set, since they are not elements of $\mathcal{E}$. Therefore, almost all trajectories of the closedloop system converge to the largest invariant set in $\mathcal{E}$, which is the location of the global minimum of $\varphi(\boldsymbol{x})$.

Before we prove that the robot will avoid collisions with obstacles, we state the following two lemmas, which will be used in the proof.

Lemma IV.6. Along system trajectories that start outside the set $\mathcal{C}, \sin (\beta)$ never changes sign and converges to zero as $t \rightarrow \infty$.

Proof. We multiply the first and second equations in Eq. (23) by $\sin (\beta)$ and $-\cos (\beta)$, respectively, and add them up. Then we obtain:

$$
\begin{equation*}
\dot{\rho} \sin (\beta)-\rho \dot{\beta} \cos (\beta) \equiv \frac{d}{d t}\left(\frac{\rho}{\sin (\beta)}\right) \sin ^{2}(\beta)=0 \tag{29}
\end{equation*}
$$

Define $\rho_{0}=\rho(0)$ and $\beta_{0}=\beta(0)$. Equation (29) shows that if $\sin \left(\beta_{0}\right) \neq 0$, then $\frac{\rho}{\sin (\beta)}$ is constant along the robot's entire trajectory, and therefore we have that

$$
\begin{equation*}
\rho(t)=\frac{\rho_{0}}{\sin \left(\beta_{0}\right)} \sin (\beta(t)) \tag{30}
\end{equation*}
$$

Since $\rho(t)$ is always positive, the sign of $\sin (\beta(t))$ must be equal to the sign of $\sin \left(\beta_{0}\right)$ for all $t \geq 0$. This implies that $\sin (\beta(t))$ never changes sign. Moreover, from Eqs. (26)-(27),
we have that $\rho(t) \rightarrow 0$ as $t \rightarrow \infty$. Equation (30) then shows that $\sin (\beta(t)) \rightarrow 0$ as $t \rightarrow \infty$. This completes the proof.

Lemma IV.7. Along system trajectories that start outside the set $\mathcal{C}, \dot{\beta}$ remains bounded and converges to zero.
Proof. The function $V$ in Eq. (26) is positive semi-definite, and $\dot{V}$ in Eq. (27) is negative semi-definite. Therefore, $V$ is always bounded, which implies that $\tan (\beta / 2)$ remains bounded. Also, using the trigonometric identity $\sin (\beta)=\frac{2 \tan \left(\frac{\beta}{2}\right)}{1+\tan ^{2}\left(\frac{\beta}{2}\right)}$ and Lemma IV.6, we can conclude that $\tan (\beta(t) / 2) \rightarrow 0$ as $t \rightarrow \infty$. This implies that $\lim _{t \rightarrow \infty} \tan (\beta(t) / 2)$ exists and is finite. Moreover, the time derivative of $\tan (\beta(t) / 2)$ is:

$$
\begin{equation*}
\frac{d}{d t}(\tan (\beta / 2))=\frac{\dot{\beta}}{2}\left(1+\tan ^{2}(\beta / 2)\right) \tag{31}
\end{equation*}
$$

By applying Barbalat's Lemma [35] to the function $\tan (\beta / 2)$, we can conclude that $\lim _{t \rightarrow \infty}\left(\frac{d}{d t}(\tan (\beta / 2))\right)=0$. This, along with the boundedness of $\tan (\beta / 2)$ in Eq. (31), proves that $\dot{\beta}(t)$ is bounded and converges to zero as $t \rightarrow \infty$.

We now substitute the expression for $\rho(t)$ in Eq. (30) into the second equation in Eq. (23) to obtain

$$
\begin{equation*}
\mu=\frac{\sin \left(\beta_{0}\right)}{k_{v} \rho_{0}} \dot{\beta} \tag{32}
\end{equation*}
$$

Since $\dot{\beta}$ is bounded by Lemma IV.7, this equation shows that $\mu$ is bounded. This gives an upper bound on $\mu:=\|\nabla \varphi\|$ along the robot's trajectory. If $\|\nabla \varphi\|$ is infinite along the boundaries of the obstacles and the domain (if it is bounded), then the boundedness of $\mu$ guarantees the robot's clearance from these boundaries. Alternatively, if $\|\nabla \varphi\|$ is bounded and has its maximum value along these boundaries, then we must design the parameters of the function $\varphi$ such that the maximum value of $\|\nabla \varphi\|$ always exceeds the upper bound in Eq. (32). This condition guarantees the robot's clearance from the boundaries of the obstacles and the domain.

## V. Simulation Results

In this section, we validate our controllers for position control and obstacle avoidance with MATLAB ${ }^{\circledR}$ simulations of a nonholonomic robot.
Position Control: First, we simulated the position controller for 12 different initial configurations of the robot, using both possible sets of control gains (positive and negative) described in Section IV. The resulting trajectories of the robot are shown in Fig. 3. From each initial configuration, shown in black, the robot is successfully stabilized to the origin (dark green point). Note that positive control gains cause the robot to move forward along its trajectory (i.e., in the direction of the $x_{r}$ axis in Fig. 1), while negative gains cause it to move backward.
Obstacle Avoidance: We evaluated our obstacle avoidance controller by setting the potential field $\varphi$ to either a navigation function $\psi$ [37] or an attractive-repulsive potential field $U$ defined as in [17]. These types of potential fields are widely used in gradient-based control methods for obstacle avoidance, and they satisfy the requirements on $\varphi$ given in Theorem IV.5.


Fig. 3: Stabilization of a simulated nonholonomic WMR to the origin from different initial configurations, using negative controller gains $k_{v}=-0.1$ and $k_{\omega}=-0.5$ (1 to 6 ), and positive controller gains $k_{v}=0.1$ and $k_{\omega}=0.5$ (7 to 12). The robot's heading at each initial configuration is indicated by a green arrow.

For the case $\varphi=\psi$, we simulated a scenario in which a robot moves to the target position, defined as the origin, through a circular environment with a radius of 4.5 m (navigation functions are defined on bounded domains) while avoiding collisions with 9 circular obstacles. We tested the controller for 7 different initial robot configurations. For the case $\varphi=U$, we simulated the same scenario and initial robot configurations, but in an unbounded environment rather than a bounded one. Figures 4 a and 4 b plot the resulting trajectories of the robot for the cases $\varphi=\psi$ and $\varphi=U$, respectively. The trajectories show that in all cases, the robot successfully avoids collisions with the obstacles as it drives to the origin.

## VI. Experimental Implementation and Results

We also implemented our controllers for position control and obstacle avoidance on a commercial Robot Operating System (ROS)-compatible nonholonomic WMR, the TurtleBot3 Burger robot produced by Robotis ${ }^{\circledR}$ (see Fig. 1). This platform is a differential-drive robot with a nonholonomic constraint on its velocity; it cannot produce any controlled motion along the direction of its wheels' axis, shown by the red dashed line in Fig. 1b. The robot uses only its onboard sensor measurements to estimate $\boldsymbol{\xi}(t)$, its configuration in the global coordinate frame; no external localization system, such as an overhead camera and vision-based tracking software, is used for this purpose. The robot can use both odometry calculations, based on the measurements of its wheel encoders, and IMU sensor data to estimate its instantaneous configuration with respect to its initial configuration in the global frame, $\boldsymbol{\xi}(0)$. Fusion of the odometry and IMU sensor data improves the accuracy of this estimate. Since we specify that the robot knows $\boldsymbol{\xi}(0)$, it can calculate its configuration $\boldsymbol{\xi}(t)$ in the global frame at any time $t>0$ by adding the sensor measurements to $\boldsymbol{\xi}(0)$. The robot's heading angle measurement is recorded in quaternion


Fig. 4: Stabilization of a simulated nonholonomic WMR to the origin from different initial configurations, while avoiding collisions with obstacles, using our proposed controller with the potential field $\varphi$ defined as (a) a navigation function [36]; (b) an attractive-repulsive potential field [17].
representation, and thus can be directly input to our proposed controllers in Eq. (7), making them convenient to implement. A video recording of the experiments described here, along with additional position control and obstacle avoidance experiments, is available online at [38].

Position Control: We implemented our position controller on the robot with both positive and negative gains and tested it for different initial robot configurations. Figures 5a and 5b display the robot's trajectory during four of the experiments, and Fig. 6 plots the time evolution of the robot's distance to the target position (the origin) during each of these experiments. The figures show that the controller stabilizes the robot smoothly to the target position from each initial configuration, without producing oscillations or cusps in the robot's trajectory.

(a) $k_{v}=0.5, k_{\omega}=1.5$. Top: scenario 1; Bottom: scenario 2.

(b) $k_{v}=0.5, k_{\omega}=1.5$. Top: scenario 1 ; Bottom: scenario 2.

Fig. 5: Experimental validation of our position controller with (a) negative controller gains, producing backward motion, and (b) positive controller gains, producing forward motion. The robot's heading at its initial configuration is indicated by a green arrow, and its trajectory is plotted as a yellow dashed line.


Fig. 6: Time evolution of $\rho$, the robot's distance to the target position, during the experimental tests of position stabilization in Fig. 5.

Obstacle Avoidance: We also implemented our obstacle avoidance controller with the potential field $\varphi$ defined as either a navigation function $\psi$ or an attractive-repulsive potential field $U$. For both cases, we tested the controller in an environment with 6 circular obstacles. In the case where $\varphi=\psi$, we defined a virtual circular boundary with a radius of 2.5 m around the obstacles, and the robot was given the centers and radii of the obstacles and boundary a priori. In the case where $\varphi=U$, the robot must be able to measure its distance from the obstacles, which we emulated by giving the robot


Fig. 7: Experimental validation of our obstacle avoidance controller, with the potential field $\varphi$ defined as (a) a navigation function [36]; (b) an attractive-repulsive potential field [17]. The robot's initial heading is indicated by a green arrow, and its trajectory is plotted as a yellow dashed line.
the same a priori information about the obstacles. Figures 7a and 7 b show the robot's trajectory during the experiment for each case. In both experiments, the robot converges to the target location (yellow circle) while avoiding collisions with the obstacles, and for the case where $\varphi=\psi$, it stays within the domain boundary (blue line in Fig. 7a). Figure 8 plots the time evolution of $\varphi(\boldsymbol{x})$ and $\|\nabla \varphi(\boldsymbol{x})\|$ at the robot's location $\boldsymbol{x}$ along its trajectory during each experiment. As expected, the values of these functions converge to zero as the robot approaches the target point. We note that in the case where $\varphi=\psi$, the robot may navigate between the obstacles (instead of circumventing them, as in Fig. 7a) if the target point is located among or close to the obstacles, as shown in the videos of additional obstacle avoidance experiments in [38]. In general, the robot's trajectory can be varied by tuning the parameters of the potential field.

## VII. CONCLUSION

In this paper, we have modified gradient-based controllers designed for holonomic WMRs to enable their implementation on nonholonomic WMRs, in order to achieve collision-free position control in environments with and without obstacles. Our controllers guarantee obstacle avoidance under the same assumptions (e.g., prior information about obstacles) as the corresponding controllers for holonomic robots, and do not impose any additional requirements. We designed a smooth,


Fig. 8: Time evolution of $\varphi$ and $\|\nabla \varphi\|$, where $\varphi=\psi$ or $\varphi=U$, during the experimental tests of obstacle avoidance in Fig. 7.
continuous, nonlinear controller that depends on trigonometric functions of the robot's heading in quaternion form, and thus does not produce undesired transient behaviors during navigation that are exhibited by discontinuous and time-varying controllers which use heading measurements in the Euler angle representation. Any gradient-based obstacle avoidance method that is developed for holonomic robots can be integrated into our controller for nonholonomic robots. We proved that the proposed controllers are stable for two sets of positive and negative control gains and demonstrated their effectiveness through simulations and experiments with a nonholonomic WMR. Future work includes designing controllers for stabilizing the pose (both position and orientation) of a nonholonomic robot while preventing collisions with obstacles using approaches other than gradient-based methods.

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[^1]:    ${ }^{1}$ See Remark IV. 4 for a discussion of particular cases in which convergence to the target position is not guaranteed.

