Single Degree of Freedom Model for Thermoelastic Damping

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Finding the thermoelastic damping in a vibrating body, for the most general case, involves the simultaneous solving of the three equations for displacements and one equation for temperature (called the heat equation). Since these are a set of coupled nonlinear partial differential equations there is considerable difficulty in solving them, especially for finite geometries. This paper presents a single degree of freedom (SDOF) model that explores the possibility of estimating thermoelastic damping in a body, vibrating in a particular mode, using only its geometry and material properties, without solving the heat equation. In doing so, the model incorporates the notion of "modal temperatures," akin to modal displacements and modal frequencies. The procedure for deriving the equations that determine the thermoelastic damping for an arbitrary system, based on the model, is presented. The procedure is implemented for the specific case of a rectangular cantilever beam vibrating in its first mode and the resulting equations solved to obtain the damping behavior. The damping characteristics obtained for the rectangular cantilever beam, using the model, is compared with results previously published in the literature. The results show good qualitative agreement with Zener's well known approximation. The good qualitative agreement between the predictions of the model and Zener's approximation suggests that the model captures the essence of thermoelastic damping in vibrating bodies. The ability of this model to provide a good qualitative picture of thermoelastic damping suggests that other forms of dissipation might also be amenable for description using such simple models. [DOI: 10.1115/1.2338054]

1 Introduction

In most solids, the strain and temperature fields are coupled. When temperature is changed, volume changes, when volume is changed by elastic deformation, temperature changes. The constant that relates the change of length (strain) with the change in temperature of a material is its thermal expansion coefficient α. When a body is elastically deformed (with volume change), thus increasing its potential energy, and is allowed to oscillate freely, the body gradually loses its potential energy and returns to its stable equilibrium even if it does not exchange energy with the environment, for example, by air drag or friction. One fundamental mechanism responsible for this dissipation is known as thermoelastic damping, wherein potential energy is converted to heat. If the body is thermally isolated from its surroundings thermoelastic damping leads to an increase in its temperature.

Thermoelastic damping can be understood from two different, but equivalent standpoints. One method is to view it as a process of dissipation of mechanical energy. In general, the stress field in a vibrating body is nonuniform and hence some regions become hotter relative to others due to thermoelastic coupling. This results in heat flow within the body, if it has finite thermal conductivity, k. Due to this heat flow, the temperature field created by thermoelastic effect in a vibrating body becomes out of phase with the stress field. Thus the temperature induced strain field is out of phase with the stress field. This phase difference between stress and strain fields leads to dissipation of mechanical energy. If k is zero, the stress and temperature fields are always in phase and hence no dissipation takes place. If k is very large, the body remains isothermal and again there is no dissipation. The second

way is to visualize thermoelastic damping is in terms of generation of entropy. Due to inhomogeneities in the stress field, local temperature gradients are created in the body. This leads to irreversible heat flow until the temperature, T, becomes uniform throughout the body, i.e., the attainment of thermal equilibrium. Since thermal equilibrium corresponds to the state of maximum entropy, there has to be a net increase in the entropy, S, of the body during this process. This increase in entropy has to come at the cost of the potential (strain)/kinetic energy of the system, since the total energy, U, of the system remains constant. The entropy increase can also be considered as an increase in heat content of the body.

The thermoelastic damping in a vibrating body can be obtained, for the most general case, by simultaneously solving the three equations for displacements and one equation for temperature (the heat equation) which comprise the equations of thermoelasticty [1]. Since these are a set of coupled nonlinear partial differential equations there is considerable difficulty in solving them, especially for finite geometries. Over the years, analytical solutions to thermoelastic damping have been obtained for certain simple geometries. Zener first studied thermoelastic relaxation as a source of damping in mechanical systems using the "standard model" of an anelastic solid [2] and developed a general theory of thermoelastic damping in a series of papers [3–5] in the 1930s. He showed that the damping behavior of transversely vibrating cantilever beam can be well approximated by a single relaxation peak with a characteristic relaxation time τ. He further showed that τ, which gives a measure of the time needed for temperature equalization through diffusion, is proportional to b^2/χ, where b is the thickness of the beam and χ is the thermal diffusivity.

Alblas [6] developed a generalised theory for thermoelastic dissipation in vibrating bodies using the three dimensional thermoelastic equations and derived the solution for the coupled thermoelastic equations in terms of normalized orthogonal eigen functions. In a later publication [7], Alblas generalized the results and obtained explicit expressions for thermoelastic damping in vibrating elastic beams, including the circular rod and the rectangular beam. Chadwick [8] derived the coupled equations govern-

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ing the thermoelastic behavior of thin plates and beams, and demonstrated that these reduce to the classical equation of motion and of heat conduction in the limit of zero coupling. Lord and Shulman [9] derived a generalized theory of thermoelasticity by considering a modified form of the Fourier heat conduction equation that took into account the time lag needed to establish steady state conduction.

In recent times, there has been renewed interest in thermoelastic damping, especially due to its contribution to energy dissipation in micromechanical resonators. High frequency micromechanical resonators, with potential applications as RF filters [10], charge detectors [11], and sensors [12,13], need to have very little energy dissipation or very high quality factor, Q. Since thermoelastic damping is a fundamental damping mechanism, it imposes an upper limit on the quality factor that can be obtained in any oscillator. Lifshitz et al. [14] evaluated the importance of thermoelastic damping at micro and nano scales and concluded that it remains relevant at these scales. Further, they derived an exact expression for thermoelastic damping in thin rectangular beams which compared favorably with Zener's well known approximation [2]. Photiadis et al. [15] proposed a simple model of thermoelastic dissipation assuming that the energy loss occurred only due to dissipation of the flexural component of motion. Houston et al. [16,17] used this model to predict thermoelastic dissipation in a single-crystal double paddle oscillator and found that the predictions agreed well with experimental observations at high temperatures (above 150 K). Nayfeh et al. [18] derived analytical expressions for quality factors in microplates, using perturbation method, by decoupling the heat equation from the equation of motion. In a recent work, Norris et al. [19] presented a general method of calculating thermoelastic damping in vibrating elastic solids, by treating elasticity as an uncoupled forcing term in the heat equation. Using this method the authors obtained a new equation of motion for flexural vibration of thin plates incorporating a thermoelastic damping term.

As mentioned earlier, the difficulty in solving thermoelastic equations, and hence finding the damping, arises because of the need to solve the displacements and heat equation simultaneously. In this paper, we explore the possibility whether thermoelastic damping in a vibrating body can be estimated based on its material properties and geometry only and without solving the heat equation. Towards the end, we introduce the notion of a modal temperature, similar to modal displacements that are often used to study vibrations of continuous systems. The rationale for introducing a modal temperature is as follows. The local strain and its rate of change, in the presence of thermoelastic coupling, determine the local temperature gradients and the rate of temperature change. Since the strain field and its rate of change can be captured using a modal displacement and the corresponding modal frequency, it might be possible to find a modal temperature that describes the temperature field in the body. This model assumes no heat flow in or out of the body and that the thermal gradients created by the strain field in one direction are much larger compared to the other two. While these assumptions may seem quite restrictive, most practical structures used in micromechanical thin beams and plates obey these assumptions [14,18].

In Sec. 2, we outline a simple spring-mass model for thermoelastic damping, and show that it exhibits damping characteristics similar to those observed in real systems. We then describe a more generalized SDOF model for thermoelastic damping and outline a procedure to derive its governing equations in Sec. 3. In Sec. 4 we use this model to find the thermoelastic damping in a vibrating cantilever beam. In Sec. 5 we compare the results obtained from the model with those previously obtained in the literature and discuss the reasons for differences between them. In the final section we discuss as to how this SDOF model can be adapted to other geometries and potentially to model other forms of dissipation.
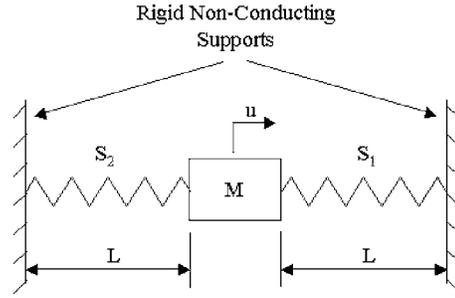
Fig. 1   Schematic of spring-mass model for thermoelasticity

2   Idealized Spring-Mass Model for Thermoelasticity

The spring-mass model (Fig. 1) comprises of a single mass M attached to two identical springs. The springs, labeled S_1 and S_2, are attached to rigid supports at their other ends. The springs are simply two elastic bars of length L, with elastic modulus E and cross-sectional area A and hence are of stiffness $K=AE/L$. The supports are assumed to be nonconducting and there is no heat transfer from the springs to the surroundings and vice versa. The springs are assumed to have very large thermal conductivity due to which the temperature within them is uniform at all times. In all the analyses below, the following initial conditions are imposed, i.e., at $t=0$:

(1) The displacement, u, of the mass satisfies, $u(0)=u_0$ and $du/dt|_{t=0}=0$;
(2) the entire system is at a uniform temperature T_0.

2.1   Spring-Mass Model with Nonconducting Mass. When the mass is displaced by a distance u ($u \ll L$), the strain in S_1, ϵ_1, is $-u/L$, and the strain in S_2, ϵ_2, is u/L. These strains induce a change in the temperature of the two springs because of thermoelastic coupling. It is assumed that the strains are uniform in each of the springs and that there is no thermal contact between the two springs, i.e., the mass M has zero thermal conductivity.

The generalized heat equation in the presence of thermoelastic coupling for an isotropic solid is given by [20]

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_v} \nabla^2 T - \frac{E \alpha T}{(1-2\nu)\rho c_v} \frac{\partial}{\partial t} \sum \epsilon_{kk} \tag{1}$$

where k is the thermal conductivity, ρ is the density, c_v is the specific heat capacity at constant volume, E is the Young's modulus, α is the linear thermal expansion coefficient, ν is the Poisson's ratio, and ϵ_{kk} are the normal components of strain.

When a body is subject to uniaxial stress (σ_{xx}), the equation reduces to

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_v} \nabla^2 T - \frac{E \alpha T}{\rho c_v} \frac{\partial \epsilon_{xx}}{\partial t} \tag{2}$$

If the thermal conductivity, k, is zero, the equation further reduces to

$$\frac{\partial T}{\partial t} = - \frac{E \alpha T}{\rho c_v} \frac{\partial \epsilon}{\partial t} \tag{3}$$

Here, ϵ_{xx} has been replaced by ϵ. Equation (3) can be viewed as a relation connecting heat generation (or absorption) rate with the strain rate for the case of uniaxial stress.

If the strain ϵ has no spatial variation, as is assumed for the springs in the spring-mass model, the partial time derivatives in Eq. (3), can be replaced by total derivatives. Hence Eq. (3) reduces to

$$\frac{dT}{dt} = -\frac{E\alpha T}{\rho c_v}\frac{d\epsilon}{dt} \tag{4}$$

Since the change in temperature due to thermoelastic effect is very small compared to the initial temperature T_0, T on the right hand side of Eq. (4) can be replaced by T_0. Integrating Eq. (4), after this modification, we get

$$\theta = -\frac{E\alpha T_0}{\rho c_v}\epsilon + N \tag{5}$$

where $\theta = T - T_0$, is the change in temperature from the initial temperature T_0 at the current strain ϵ, $N = (E\alpha T_0/\rho c_v)\epsilon_0$ is the constant of integration and ϵ_0 is the strain at $t = 0$.

When S_1 is subjected to a strain $-u/L$, its temperature increases by $\theta_1 = E\alpha T_0 u/\rho c_v L + N_1$. Due to thermal expansion, the length of the spring increases by $\Delta L_1 = \alpha\theta_1 L = E\alpha^2 T_0 u/\rho c_v + \alpha N_1 L$. In effect, the net compression experienced by S_1 is $u(1 + E\alpha^2 T_0/\rho c_v) + \alpha N_1 L$. Hence, the force exerted by S_1 on the mass M is given by $F_1 = -K(u(1 + E\alpha^2 T_0/\rho c_v) + \alpha N_1 L)$. Similarly, the force exerted by S_2 is $F_2 = -K(u(1 + E\alpha^2 T_0/\rho c_v) - \alpha N_2 L)$. The initial conditions, $\theta_1 = \theta_2 = 0$ and $u = u_0$ at $t = 0$ can be used to find N_1 and N_2.

Hence, the equation of motion of the spring-mass system is given by

$$M\frac{d^2u}{dt^2} + 2K\left(1 + \frac{E\alpha^2 T_0}{\rho c_v}\right)u + K(\alpha N_1 L - \alpha N_2 L) = 0 \tag{6}$$

which is of the form

$$a'\frac{d^2u}{dt^2} + b'u + c' = 0 \tag{7}$$

and has the general solution $u = -c'/b' + P_1\cos(\omega t) + P_2\sin(\omega t)$ where $\omega = \sqrt{b'/a'}$. On imposing the initial conditions the solution reduces to $u = -c'/b' + P\cos(\omega t)$ where P is determined by the condition $u = u_0$ at $t = 0$.

The solution to the equation of motion of the discrete spring-mass system with zero thermal conductivity leads to the following conclusions:

(1) The motion is harmonic with no decay in amplitude, i.e., with no damping, as is the case with any continuous system that has zero thermal conductivity;
(2) the natural frequency, $\sqrt{2K(1 + E\alpha^2 T_0/\rho c_v)/M}$ of the thermoelastically coupled system is higher than $\sqrt{2K/M}$, the frequency of the uncoupled system ($\alpha = 0$), again a characteristic of any continuous system;
(3) the point at which the mass experiences zero force is shifted from $u = 0$, of the uncoupled system, to $u = c'/b' \neq 0$.

2.2 Spring-Mass Model with Conducting Mass. In this analysis, the mass M is assumed to have a finite thermal conductivity which leads to heat flow between the springs when there is a finite temperature difference between them. The rate of heat flow between the springs is assumed to be proportional to the difference in temperature between the springs, i.e., $\dot{q} = kL_c(T_1 - T_2) = kL_c(\theta_1 - \theta_2)$ (since $T_1 = T_0 + \theta_1$ and $T_2 = T_0 + \theta_2$). Here k is the thermal conductivity of the mass and L_c is a parameter that determines the heat transfer across it.

In a small time interval dt, let du be the displacement of the mass and $d\theta_1$ and $d\theta_2$ be the change in temperature in S_1 and S_2 (Fig. 1). The change in force exerted by S_1 and S_2 on the mass is given by

$$dF_1 = -K(du + L\alpha\,d\theta_1) \tag{8}$$

$$dF_2 = -K(du - L\alpha\,d\theta_2) \tag{9}$$

Dividing by dt throughout, and using $dF/dt = Md^3u/dt^3$ the equation of motion of the spring-mass system becomes

$$M\frac{d^3u}{dt^3} + 2K\frac{du}{dt} + KL\alpha\left(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}\right) = 0 \tag{10}$$

In Eq. (10), the masses of the springs (M_s) have been neglected since they are very small compared to M. To obtain the expressions for θ_1 and θ_2, the heat balance equation needs to be used. In the absence of thermal conductivity the relation governing the rate of temperature change as a function of the strain rate is given by Eq. (4). Since, in the spring-mass system K is analogous to E, M_s/L is analogous to ρ and u is analogous to ϵ, Eq. (4) for the mass-spring system becomes

$$\frac{dT}{dt} = \frac{d\theta}{dt} = -\frac{K\alpha LT}{M_s c_v}\frac{du}{dt} \tag{11}$$

If there is heat transfer, the heat transfer rate given by $\dot{q} = kL_c(\theta_1 - \theta_2)$ has to be added to Eq. (11). Hence, the heat balance equations for S_1 and S_2 are given by

$$M_s c_v\frac{d\theta_1}{dt} = K\alpha LT_1\frac{du}{dt} - kL_c(\theta_1 - \theta_2) \tag{12}$$

$$M_s c_v\frac{d\theta_2}{dt} = -K\alpha LT_2\frac{du}{dt} + kL_c(\theta_1 - \theta_2) \tag{13}$$

The first term on the RHS of these two equations can be considered as the rate at which heat is generated/absorbed due to change in displacement with respect to time. Equation (10) along with Eqs. (12) and (13) describe the dynamics of the spring-mass system in the presence of thermoelastic coupling. These are a set of nonlinear differential equations and have to be solved numerically since there is no straightforward method for obtaining an analytical solution. But, since the change in temperatures, θ_1 and θ_2, induced by thermoelastic effect is very small compared to the initial temperature T_0, we can replace T_1 and T_2 in the first term on the RHS of Eqs. (12) and (13) by T_0, i.e., linearize Eqs. (12) and (13) about T_0 and consider only the zeroth order term. Therefore, we get

$$M_s c_v\frac{d\theta_1}{dt} = -M_s c_v\frac{d\theta_2}{dt} = K\alpha LT_0\frac{du}{dt} - kL_c(\theta_1 - \theta_2) \tag{14}$$

One consequence of linearization is that there will be no net increase in the temperature of the body, i.e., the net heat generated in the body will be zero, even though there is a reduction in potential energy. The linearized equations can easily be solved analytically as will be shown below.

Defining a new variable $\theta^* = \theta_1 - \theta_2$, Eqs. (10) and (14) can be reduced to

$$M\frac{d^3u}{dt^3} + 2K\frac{du}{dt} + KL\alpha\frac{d\theta^*}{dt} = 0 \tag{15}$$

$$M_s c_v\frac{d\theta^*}{dt} = 2K\alpha LT_0\frac{du}{dt} - 2kL_c\theta^* \tag{16}$$

Integrating Eq. (15) with respect to time and using the initial condition, $Md^2u/dt^2 = -2Ku$ and $\theta^*(0) = 0$, we get

$$M\frac{d^2u}{dt^2} + 2Ku + KL\alpha\theta^* = 0 \tag{17}$$

Substituting for θ^* in Eq. (16) from Eq. (17), we get

$$G_1\frac{d^3u}{dt^3} + G_2\frac{d^2u}{dt^2} + G_3\frac{du}{dt} + G_4u = 0 \tag{18}$$

where $G_1 = M_s M c_v/KL\alpha$, $G_2 = 2kL_c M/KL\alpha$, $G_3 = 2(KL\alpha T_0 + M_s c_v/L\alpha)$ and $G_4 = 4kL_c/L\alpha$ are positive constants. Since Eq. (18) is a third order ordinary differential equation with constant coefficients, we can look for solutions of the form $u = Ae^{pt}$. Substituting this in Eq. (18), we get

$$G_1 p^3 + G_2 p^2 + G_3 p + G_4 = 0 \qquad (19)$$

If p_1, p_2, and p_3 are the roots of Eq. (19), the solution for u is given by

$$u = U_1 e^{p_1 t} + U_2 e^{p_2 t} + U_3 e^{p_3 t} \qquad (20)$$

where U_1, U_2, and U_3 are determined by the initial conditions.

Since, all the coefficients in Eq. (19) are positive, p can either be real and negative or complex with negative real part. Unless the system is over damped, p_1 and p_2 will be complex with negative real parts and p_3 will be real and negative. Let $p_1 = r + is$ and $p_2 = r - is$, where r is negative and hence represents the decaying component of the solution in steady state while s represents the harmonic component. Since the damping due to thermoelastic effect is small, in general r will be much smaller in magnitude compared to s. If p_3 is comparable in magnitude to s, the transient part (the third term in the RHS of Eq. (20)) goes to zero in a small time and the system achieves steady state quickly. Using the initial conditions, $u(0) = 0$ and $\dot{u}(0) = 0$ it can be shown that $U_1 = U_2 = P$, where P is a constant, and hence the solution in steady state reduces to $u = 2Pe^{rt}\cos(st)$.

The parameter L_c, as mentioned earlier, determines the heat flow in the system and hence the damping due to thermoelastic effect. $L_c = 0$ corresponds to the adiabatic case while $L_c = \infty$ corresponds to the isothermal case, both of which lead to zero damping. For any other finite value of L_c there will be finite dissipation due to thermoelastic effect with the damping peaking at an unique value of L_c. In effect, one needs to determine L_c for the system of interest before the model can be used to predict its thermoelastic behavior. The equation of motion obtained for the case of nonconducting mass in the previous section can also be obtained from Eqs. (16) and (17) by making $L_c = 0$.

In the spring-mass model, the amplitude exhibits an exponential decay in steady state as is expected in real systems. The model implicitly incorporates the notion of a modal temperature as it models the heat flow in terms of the difference of two lumped temperatures T_1 and T_2 which represent two parts of the system that have opposite states of strain. This indicates that it might be possible to find a modal temperature for real systems.

3 Generalized Model for Thermoelastic Damping

The generalized model provides a framework for estimating the thermoelastic damping in a thermoelastically coupled system vibrating in a particular mode. In estimating the damping using this model, we make the following assumptions:

(1) The dynamics of the vibrating system can be captured by a single modal displacement and frequency;
(2) the stress and strain fields in the system and their rate of change, in the absence of thermoelastic coupling, are completely known for the mode of vibration we are interested in;
(3) the stress field remains unaltered even in the presence of thermoelastic coupling;
(4) the temperature change induced is determined by the uncoupled strain;
(5) there is no heat flow from the surroundings to the system or vice versa.

In effect, we try to incorporate the effect of thermoelastic coupling by treating it as a perturbation from the uncoupled state.

To obtain the thermoelastic damping using this model we adopt the following procedure:

(1) We partition the vibrating system into two regions, 1 and 2, that are anti-phase with respect to strain (i.e., the strains of these two regions are of opposite nature at all times) and choose one point in each of these regions to be their respective "modal points;"
(2) we take the temperature *changes* at these modal points

(from now on referred to as "modal temperatures") to represent the temperature fields of the two regions. This is the crucial approximation underlying this model as it reduces the temperature field of the distributed system to two lumped modal temperatures.
(3) we use the heat balance for the two regions and the energy conservation in the system to obtain the three governing equations necessary to solve for the three independent variables, namely the two modal temperatures and the modal displacement;
(4) finally, we use the time evolution of the modal displacement, obtained by solving the governing equations, to compute the damping in the system.

We will show that the damping characteristics obtained from the model is insensitive to the choice of the modal points. In other words, we get the same damping characteristics irrespective of the choice of modal points as long as we follow the definitions consistently.

3.1 Derivation of Governing Equations. We start by considering the the energy conservation equation, which for a general system vibrating in a particular mode, in the absence of thermoelastic coupling, is given by

$$\frac{1}{2}M_{\text{eff}}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}K_{\text{eff}}u^2 = E_0 \qquad (21)$$

where, M_{eff} is the effective mass and K_{eff} is the effective stiffness with respect to the modal displacement, u, while E_0 is the initial energy of the system. In the presence of coupling this equation modifies to

$$\frac{1}{2}M_{\text{eff}}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}K_{\text{eff}}u^2 - \frac{1}{2}\int_V \sigma_{ii}\alpha\theta\, dV + \rho c_v \int_V \theta\, dV = E_0 \qquad (22)$$

where θ is the change in temperature from the initial temperature T_0 and V is the volume of the system. The third term on the LHS of Eq. (22) accounts for the strain energy due to thermal strain while the last term accounts for the change in heat content of the system. Equation (22) can be expanded as

$$\frac{1}{2}M_{\text{eff}}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}K_{\text{eff}}u^2 - \frac{1}{2}\left(\int_{V_1}\sigma_{ii}\alpha\theta\, dV + \int_{V_2}\sigma_{ii}\alpha\theta\, dV\right)$$
$$+ \rho c_v\left(\int_{V_1}\theta\, dV + \int_{V_2}\theta\, dV\right) = E_0 \qquad (23)$$

where V_1 and V_2 are the volumes of the two regions, 1 and 2, and hence $V_1 + V_2 = V$.

To proceed further, we consider a mode where only one stress component, σ, contributes most of the strain energy. As we are interested in solving for the modal displacement, u, and the modal temperatures, θ_1 and θ_2, we need to formulate the governing equations in terms of u, θ_1, and θ_2. Towards this end, we replace the integrals in Eq. (23) in terms of θ_1, θ_2 and σ_1, σ_2, the stresses at the modal points, by defining

$$\rho c_v \int_{V_i}\theta\, dV = A_i\theta_i \quad i = 1,2 \qquad (24)$$

$$\frac{1}{2}\int_{V_i}\sigma\alpha\theta\, dV = B_i\sigma_i\alpha\theta_i \quad i = 1,2 \qquad (25)$$

where A_i and B_i are constants. Equation (23), hence, becomes
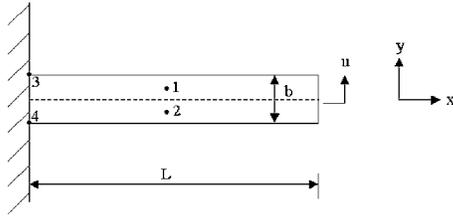
Fig. 2 Schematic of cantilever beam

$$\frac{1}{2}M_{\text{eff}}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}K_{\text{eff}}u^2 - (B_1\sigma_1\alpha\theta_1 + B_2\sigma_2\alpha\theta_2) + A_1\theta_1 + A_2\theta_2 = E_0$$
(26)

Next, we consider the heat balance for the two regions represented by θ_1 and θ_2. The heat balance equations, assuming that at any time both θ_1 and θ_2 are much smaller than T_0, will be be similar to Eq. (14) and are given by

$$A_1\frac{d\theta_1}{dt} = -A_2\frac{d\theta_2}{dt} = H\frac{du}{dt} - kL_c(\theta_1 - \theta_2)$$
(27)

where H is the constant relating the rate of heat generation/absorption in the two regions to the modal displacement (strain) rate. Equation (27) implies that the net heat generated in the system is zero which, as mentioned earlier, is a consequence of considering the linearized heat equation. Differentiating Eq. (26) with respect to time and noting that both σ_1, σ_2 are proportional to u, i.e., $\sigma_i = C_i u$, $i = 1, 2$ (C_i are constants), we get, using Eq. (27)

$$M_{\text{eff}}\left(\frac{d^2u}{dt^2}\right)\left(\frac{du}{dt}\right) + K_{\text{eff}}u\frac{du}{dt} - D_1\alpha\left(\frac{du}{dt}\theta_1 + \frac{d\theta_1}{dt}u\right) - D_2\alpha\left(\frac{du}{dt}\theta_2\right.$$
$$\left. + \frac{d\theta_2}{dt}u\right) = 0$$
(28)

where $D_i = B_i C_i$.

Equations (27) and (28) define the dynamics of the system in the presence of thermoelastic coupling. We can find the solution to these equations in the steady state by assuming, $u = u_0 e^{i\omega t}$ and $\theta_1 = \theta_{10} e^{i\omega t}$, $\theta_2 = \theta_{20} e^{i\omega t}$. In general, ω will be complex with $\text{Re}(\omega)$ giving the new eigen frequency in the presence of thermoelastic damping and $|\text{Im}(\omega)|$ giving the attenuation of vibration. A measure of thermoelastic damping is then be given by [14]

$$Q^{-1} = 2\left|\frac{\text{Im}(\omega)}{\text{Re}(\omega)}\right|$$
(29)

where Q^{-1} is the inverse of the quality factor.

Equation (27) involves a system parameter L_c which governs the heat flow within the system. L_c, which in general will depend on the geometry of the system, determines the lag between the temperature and stress fields and hence the dissipation. For certain simple cases we can get a good estimate of L_c by considering the heat flow in the system. But for more complicated geometries, it might have to be deduced from experiments. In Sec. 5, we provide an intuitive picture of L_c and outline a method for estimating it.

4 Application of Generalized Model to Vibrating Cantilever Beam

To estimate the thermoelastic damping in a thin cantilever beam vibrating in its first mode using the model, we first consider the small flexural displacements of the cantilever beam in the absence of thermoelastic coupling. The cantilever beam is taken to be of length L, thickness $b(\ll L)$ and width $w(\ll L)$ (Fig. 2). The x axis is defined to be parallel to the beam axis and y and z axes are parallel to surfaces with dimensions b and w, respectively. Since $b \ll L$ and $w \ll L$, it can be assumed that any plane cross section,

initially perpendicular to the beam axis, remains perpendicular to the neutral plane during bending. If the surfaces of the beam are assumed to be stress free, only the σ_{xx} component of stress will be present. Since b is small (thin beam) this assumption holds good in the interior as well. Under these assumptions it can be shown that [20]

$$\epsilon_{xx} = -y\frac{\partial^2 v}{\partial x^2}$$
(30)

and the equation of motion is given by [20]

$$\rho A\frac{\partial^2 v}{\partial t^2} + EI\frac{\partial^4 v}{\partial x^4} = 0.$$
(31)

where $A = bw$ is the area of its cross section, $I = b^3 w/12$ is the moment of inertia about the z axis and $v(x,t)$ is the displacement of the beam along y direction at time t.

If the cantilever is given an initial displacement along y direction, such that its shape matches with its first mode shape and the tip displacement is u_0, and set into vibration, $v(x,t)$ will take the form $v(x,t) = v_0(x)\cos(\omega t)$, where $v_0(x)$ is the mode shape and ω is the natural frequency of the cantilever. The general solution for $v_0(x)$ is given by

$$v_0(x) = B(\sin(qx) - \sin h(qx) + R(\cos(qx) - \cos h(qx)))$$
(32)

where $R = (\cos(qL) + \cos h(qL))/(\sin(qL) - \sin h(qL))$ and B is determined by the initial displacement u_0 and is approximately equal to $u_0/2.724$. Since we are considering only the first mode, $q = q_1 \approx 1.875/L$ and $R \approx 1.362$. Hence $v(x,t)$, for the first mode, is given by

$$v(x,t) = \frac{u_0\cos(\omega t)}{2.724}(\sin(q_1 x) - \sin h(q_1 x) + R(\cos(q_1 x)$$
$$- \cos h(q_1 x)))$$
(33)

As $u_0\cos(\omega t) \equiv u(t)$, where $u(t)$ is the displacement of the cantilever tip, Eq. (33) can be written as

$$v(x,t) = \frac{u(t)}{2.724}(\sin(q_1 x) - \sin h(q_1 x) + R(\cos(q_1 x) - \cos h(q_1 x)))$$
(34)

We can find M_{eff} of the cantilever with respect to u by equating the kinetic energy of the cantilever to $M_{\text{eff}}(du/dt)^2/2$. K_{eff} can similarly be found by equating the potential energy to $K_{\text{eff}}u^2/2$. M_{eff} and K_{eff}, the modal mass and stiffness thus found, are given by $M_{\text{eff}} \approx 0.25M$ and $K_{\text{eff}} \approx 0.2575Eb^3 w/L^3$. $M = \rho bLw$ is the total mass of the cantilever.

To proceed further, we choose the points 1 and 2, as shown in Fig. 2, to be the modal points and designate the temperature *changes* at these points, θ_1 and θ_2, to be the modal temperatures. The modal points have been chosen symmetrically merely for convenience. From Eqs. (24) and (25) we get

$$\rho c_v \int_0^L \int_0^{b/2} \int_{-w/2}^{w/2} \theta\,dz\,dy\,dx = A_1\theta_1$$
(35)

$$\rho c_v \int_0^L \int_{-b/2}^0 \int_{-w/2}^{w/2} \theta\,dz\,dy\,dx = A_2\theta_2$$
(36)

$$\frac{1}{2}\int_0^L \int_0^{b/2} \int_{-w/2}^{w/2} \sigma\alpha\theta\,dz\,dy\,dx = B_1\sigma_1\alpha\theta_1$$
(37)

$$\frac{1}{2}\int_0^L \int_{-b/2}^0 \int_{-w/2}^{w/2} \sigma\alpha\theta\, dz\, dy\, dx = B_2\sigma_2\alpha\theta_2 \qquad (38)$$

As H is the constant relating the rate of heat generation/absorption in the two regions to the modal displacement rate, we have

$$\rho c_v \int_0^L \int_0^{b/2} \int_{-w/2}^{w/2} \frac{\partial\theta}{\partial t} dz\, dy\, dx = H\frac{du}{dt} \qquad (39)$$

To calculate the constants A_i, B_i, and H exactly we need to know the actual temperature profile, which can be obtained only by solving the heat equation for the system under consideration with appropriate boundary conditions. Since the very purpose of the model is to get an estimate of damping without solving the heat equation, we make the following approximation. We solve for the temperature profile for the case of no heat flow ($k \to 0$), so that the temperature profile is solely determined by the strain, and use it to determine the constants. As the stress field is also known, we can compute the constants A_1, A_2, B_1, B_2, and H. The values of the constants are $A_1 = A_2 = 0.5\ \rho c_v bLw$, $B_1 = B_2 = 0.5445\ wbL$, and $H = 0.172\ E\alpha T_0 wb^2/L$.

Using the stress field, we find the constants C_1 and C_2 that relate σ_1 and σ_2 with u to be $C_1 = -C_2 = -0.344Eb/L^2$. With these constants, together with M_{eff} and K_{eff} we can obtain the equations that govern the dynamics of the cantilever in the presence of thermoelastic coupling from Eqs. (27) and (28). The equations are given by

$$0.5Mc_v\frac{d\theta_1}{dt} = -0.5Mc_v\frac{d\theta_2}{dt} = 0.172\frac{E\alpha T_0 wb^2}{L}\frac{du}{dt} - kL_c(\theta_1 - \theta_2) \qquad (40)$$

$$M_{\text{eff}}\frac{du}{dt}\left(\frac{d^2u}{dt^2}\right) + K_{\text{eff}}u\frac{du}{dt} + 0.1873\frac{E\alpha b^2 w}{L}\left(\frac{du}{dt}(\theta_1 - \theta_2)\right. \\ \left. + u\left(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}\right)\right) = 0 \qquad (41)$$

Using $\theta^* = \theta_1 - \theta_2$, Eqs. (40) and (41) reduce to

$$0.5Mc_v\frac{d\theta^*}{dt} = 0.344\frac{E\alpha T_0 wb^2}{L}\frac{du}{dt} - 2kL_c\theta^* \qquad (42)$$

$$M_{\text{eff}}\left(\frac{d^2u}{dt^2}\right)\left(\frac{du}{dt}\right) + K_{\text{eff}}u\frac{du}{dt} + 0.1873\frac{E\alpha b^2 w}{L}\left(\theta^*\frac{du}{dt} + u\frac{d\theta^*}{dt}\right) = 0 \qquad (43)$$

5 Results and Discussion

To examine the validity of the approach outlined above in modeling thermoelastic dissipation in rectangular cantilever beams, we solve Eqs. (42) and (43) and compare the results with those previously published in literature. For solving the equations we take $\alpha = 2.616 \times 10^{-6}/°C$, $c_v = 713\ J/KgK$, $k = 156\ W/mK$, $E = 1.68 \times 10^{11}\ N/m^2$, $\rho = 2330\ Kg/m^3$. Assuming a steady state solution of the form $u = u_0 e^{i\omega t}$ and $\theta^* = \theta_0^* e^{i\omega t}$, we can solve Eqs. (42) and (43) for the complex valued frequency ω and hence the damping.

Figure 3 gives the plots of the normalized attenuation, $\zeta = |\text{Im}(\omega)|/(\omega_0\Delta_E)$ and normalized frequency shift, $\beta = (\text{Re}(\omega) - \omega_0)/(\omega_0\Delta_E)$, for a beam of dimensions $L = 4 \times 10^{-4}$ m, $b = 4 \times 10^{-5}$ m, and $w = 4 \times 10^{-5}$ m, as a function of L_c, where Δ_E is the relaxation strength [2] and ω_0 is the isothermal natural frequency. The relaxation strength can be understood as follows. When a periodic stress of frequency ω is applied to a body, the stress and strain amplitudes are related through the frequency dependent complex elastic modulus. The dissipation in the body, Q^{-1}, if small, is then equal to the ratio of the imaginary and real parts of the complex modulus giving $Q^{-1}/\Delta = \omega\tau/(1 + \omega^2\tau^2)$, where Δ is
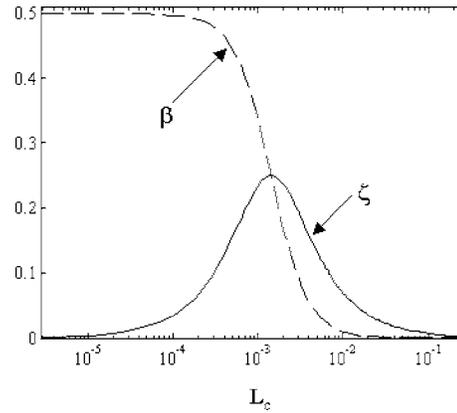
Fig. 3 Plot of normalized attenuation, ζ, and normalized frequency shift, β, as a function of L_c

the relaxation strength and τ is the relaxation time. For a thermoelastic solid the relevant relaxation strength is that of the Young's modulus giving $\Delta_E = (E_{ad} - E)/E = E\alpha^2 T_0/\rho c_v$, where E_{ad} and E are the adiabatic and isothermal values of the Young's modulus.

As can be seen from Fig. 3, the frequency shift attains a maximum when $L_c \to 0$ and goes to zero as L_c becomes larger. The damping on the other hand attains a peak for an intermediary value of L_c and tends to zero at both the extremes. Since $L_c \to 0$ represents an adiabatic system and large values of L_c represent a nearly isothermal system, it can be seen that the damping behavior and frequency shift predicted by the model agrees well with results obtained by Zener [3] and Lifshitz et al. [14].

As mentioned in Sec. 3, we need to know the L_c of a system before we can find its thermoelastic damping. One method to get an estimate of L_c is as follows. Since, we take the heat flow between the two regions to be equal to $kL_c(\theta_1 - \theta_2)$, we have

$$k\int_S \frac{\partial\theta}{\partial n} dS = kL_c(\theta_1 - \theta_2) \qquad (44)$$

where S is the surface through which heat flows between the regions represented by θ_1 and θ_2. To calculate L_c exactly from Eq. (44) we need to know the actual temperature profile. To circumvent this problem we make the same approximation that we made for evaluating the constants in Eqs. (35)–(39), i.e., solve for the temperature profile for the case of no heat flow ($k \to 0$) and use it to determine L_c. For the particular case of the vibrating cantilever that we are considering here, Eq. (44) becomes

$$k\int_0^L \int_{-w/2}^{w/2} \left.\frac{\partial\theta}{\partial y}\right|_{y=0} dz\, dx = kL_c(\theta_1 - \theta_2) \qquad (45)$$

On solving Eq. (45) we get $L_c = 2Lw/b$. It is worth noting that for the simple case of cantilever beam considered here we can get an order of magnitude of L_c based on elementary physical considerations. The heat flow rate, $\dot{q}$, between two bodies at temperatures T_1 and T_2 is given by $\dot{q} = kA(T_1 - T_2)/d$ where k is the thermal conductivity, A is the cross sectional area, and d is the length of the heat conduction path. A comparison with $\dot{q} = kL_c(\theta_1 - \theta_2)$, used in the model, reveals that L_c is analogous to A/d. The appropriate area and length for the cantilever beam are $A = Lw$ and $d = b$, which gives $L_c \propto Lw/b$.

Figure 4 shows the plots of Q^{-1}/Δ_E as a function of b/L, with $L = 4 \times 10^{-4}$ m and $w = 4 \times 10^{-5}$ m, as obtained from the model (solving Eqs. (42) and (43) with $L_c = 2Lw/b$) and from Zener's approximation (Eq. (46)). Zener's approximation for Q^{-1} is given by
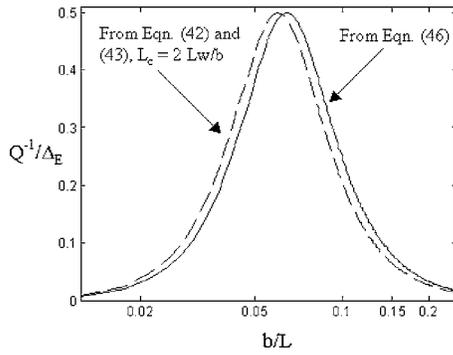
Fig. 4 Plot of Q^{-1}/Δ_E versus b/L obtained from the model and Zener's result

$$\frac{Q^{-1}}{\Delta_E} = \frac{\omega\tau}{1 + \omega^2\tau^2} \qquad (46)$$

where $\tau = b^2/\pi^2\chi$. Since $\omega \propto b/L^2$, Q^{-1}/Δ_E is just a function of b^3/L^2. In other words, if b^3/L^2 is kept constant, the damping remains constant. If we solve Eqs. (42) and (43) keeping b^3/L^2 constant, the damping remains constant, irrespective of the value of w. Further, if we choose $L_c = 2.466Lw/b$, the damping predicted by the model and Eq. (50) match exactly. These results imply that

(1) The damping predicted by the model is also just a function of b^3/L^2;
(2) the functional form of the relation between damping and b^3/L^2 predicted by the model is the same as that of Zener's approximation.

While the qualitative damping behavior predicted by the model agrees well with Zener's result, the quantitative agreement is not exact because of the approximation involved in evaluating the constants in Eqs. (35)–(39) as well as L_c. In estimating these constants and L_c, we implicitly assume that the temperature distribution within the cantilever remains similar irrespective of whether there is heat flow or not. In other words, if θ_A and θ_B represent the temperature changes at two arbitrary points in steady state, we assume θ_A/θ_B, with heat flow (k is finite) is equal to θ_A/θ_B when there is no heat flow ($k=0$). But, solving the heat equation explicitly gives a temperature distribution that depends on k, as shown in for example [14], leading to the discrepancy. This k dependence of the temperature distribution, at first sight, would suggest that at larger values of k, the damping characteristics obtained from the model will be substantially different from those obtained from Zener's result. But, the closeness between the damping characteristics obtained from the model and Zener's result remains identical for all finite values of k. This suggests that the degree of approximation is insensitive to the value of k, which we think is a consequence of averaging the temperature distribution in evaluating the constants.

As mentioned earlier, the damping characteristics obtained from the model is independent of the choice of modal points. For example, if we choose the two modal points as the corners of the upper and lower halves of the cantilever (points 3 and 4 in Fig. 2), we obtain

$$0.09785 Mc_v \frac{d\theta^*}{dt} = 0.344 \frac{E\alpha T_0 w b^2}{L}\frac{du}{dt} - 2kL_c\theta^* \qquad (47)$$

$$M_{\text{eff}}\left(\frac{d^2u}{dt^2}\right)\left(\frac{du}{dt}\right) + K_{\text{eff}} u\frac{du}{dt} + 0.03665\frac{E\alpha b^2 w}{L}\left(\theta^*\frac{du}{dt} + u\frac{d\theta^*}{dt}\right) = 0 \qquad (48)$$

where M_{eff} and K_{eff} are the same as in the previous case but $L_c = 0.3914Lw/b$ instead of $2Lw/b$. Solving Eqs. (47) and (48) leads

to exactly the same damping behavior as shown in Fig. 4.

The functional dependence of L_c on the dimensions (like length, radius, width, etc.) of the vibrating system remains the same irrespective of the size. Once this functional relationship is known, the damping in any similar system vibrating in the same mode can be easily obtained. This will be especially useful for estimating thermoelastic damping in systems with complex geometries, as the model requires only the knowledge of the stress field. For example if we need to find the thermoelastic damping of beams with U-shaped cross section vibrating in a particular mode, we need to determine the functional relationship only once, and we can use the model to predict the damping at all size scales. But, when applying the model to more complex geometries two issues need to be taken into consideration:

(1) The functional dependence of L_c on the the dimensions of the body may not be obvious, i.e., while it was easy to see that $L_c \propto L$, w and $L_c \propto 1/b$ for the rectangular cantilever beam, such dependence may not be apparent for more complex geometries;
(2) Estimating L_c by solving Eq. (44) might be more difficult.

One way to overcome these problems would be to experimentally obtain the damping for the system of interest at various size scales, and determine the functional dependence of L_c on the dimensions using it. For this method to work, though, one would have to make sure that energy dissipation due to other causes are minimal. A second method would be to obtain the damping at different size scales by numerically solving the equations of thermoelasticity and determine the relationship between L_c and the dimensions of the system.

The simple model described in this work essentially depends on finding one macroscopic system parameter, L_c, to determine the thermoelastic damping in a vibrating solid. We have shown that this model provides a reasonably good estimate of thermoelastic damping at least for the case of a rectangular cantilever beam. This raises the possibility that similar macroscopic parameters and a corresponding damping model can be used to estimate dissipation due to other relaxation processes like dislocation dynamics or grain boundary relaxation. For example, the average grain size might turn out to be a macroscopic parameter on which grain boundary relaxation depends. Finding a good damping model, though, might prove to be more difficult, as many of these relaxation processes, unlike thermoelastic damping, do not have a well developed theoretical framework or governing equations.

6 Concluding Remarks

A SDOF model for estimating thermoelastic damping in structures vibrating in specific modes is proposed in this work. The model, incorporating the notion of modal temperatures, estimates the thermoelastic damping in a structure without explicitly solving the heat equation. The good qualitative agreement between the predictions of the model and Zener's well known approximation, for the rectangular cantilever beam, suggests that the model captures the essence of thermoelastic damping in vibrating bodies. The model can be used to estimate thermoelastic damping in structures with more complicated geometries if their corresponding characteristic length, L_c, can be obtained.

The simplicity of the model rests on several assumptions: (a) The dynamics of the vibrating body can be captured by a single modal displacement and frequency, (b) the stress and strain fields are completely known for the uncoupled state, (c) the dynamic response of the body in the presence of thermoelastic coupling differs only slightly from the uncoupled state, and (d) there is no thermal interaction between the body and the environment. Finally, the ability of this model to provide a good qualitative picture of thermoelastic damping suggests that other forms of dissipation might also be amenable for description using such simple models.

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