Multicast Capacity of Packet-Switched Ring WDM Networks

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***Abstract*—Packet-switched unidirectional and bidirectional ring wavelength division multiplexing (WDM) networks with destination stripping provide an increased capacity due to spatial wavelength reuse. Besides unicast traffic, future destination stripping ring WDM networks also need to support multicast traffic efficiently. This article examines the largest achievable transmitter throughput, receiver throughput, and multicast throughput of both unidirectional and bidirectional ring WDM networks with destination stripping. A probabilistic analysis evaluates both the nominal capacity, which is based on the mean hop distances traveled by the multicast packet copies, and the effective capacity, which is based on the ring segment with the highest utilization probability, for each of the three throughput metrics. The developed analytical methodology accommodates not only multicast traffic with arbitrary multicast fanout but also unicast and broadcast traffic. Numerical investigations compare the nominal transmission, receiver, and multicast capacities with the effective transmission, receiver, and multicast capacities and examine the impact of number of ring nodes and multicast fanout on the effective transmission, reception, and multicast capacity of both types of ring networks for different unicast, multicast, and broadcast traffic scenarios and different mixes of unicast and multicast traffic. The presented analytical methodology enables the evaluation and comparison of future multicast-capable medium access control (MAC) protocols for unidirectional and bidirectional ring WDM networks in terms of transmitter, receiver, and multicast throughput efficiency.**

***Index Terms*—Average hop distance, destination stripping, multicast, ring network, spatial wavelength reuse, wavelength division multiplexing (WDM).**

I. Introduction

PACKET-switched ring wavelength division multiplexing (WDM) networks have received significant interest in recent years as solutions to the emerging capacity shortage in the metropolitan area [1]. Typically, these ring networks deploy *destination stripping*. With destination stripping the intended destination nodes pull data packets from the ring, as opposed to source stripping rings where the source nodes remove the packets from the ring after one round-trip time. With destination stripping, nodes downstream from the destination nodes are able to spatially reuse wavelengths. Due to spatial wavelength reuse, multiple transmissions can take place simultaneously on each of the wavelength channels, resulting in an increased network capacity. For uniform unicast traffic, where each of the N ring nodes generates the same amount of traffic and a given packet is destined to any of the remaining $(N-1)$ ring nodes (aside from the packet's source node) with equal probability $1/(N-1)$, the mean hop distance that a packet travels on a unidirectional ring is $N/2$. Thus, on average up to two transmissions can take place on each wavelength at the same time, resulting in a stability limit (in terms of the maximum number of simultaneously ongoing transmissions) of twice the number of wavelength channels. We refer to this stability limit which is based on the mean hop distance as *nominal capacity*. On a bidirectional ring with spatial wavelength reuse and shortest path steering, a packet travels a mean hop distance of approximately $N/4$, resulting in a nominal capacity of approximately four times the number of wavelength channels.

Besides unicast traffic, for which the mean hop distances and resulting nominal capacity can be calculated relatively easily, future ring WDM networks are expected to transport a significant portion of *multicast* (multidestination) traffic, arising from applications such as distributed games, video conferencing, and distance learning. To date, multicasting has received a great amount of attention in broadcast-and-select (see, e.g.,[2]–[4]), wavelength-routed (see, e.g.,[5]–[7]), and optical burst-switched (OBS) WDM networks (see, e.g.,[8]), but only little attention in ring WDM networks [9]. Most reported medium access control (MAC) protocols for ring WDM networks aim at improving the throughput-delay performance and providing fairness and/or quality of service (QoS) primarily for unicast traffic [1]. Future MAC protocols for ring WDM networks also need to support multicast traffic efficiently. To enable the assessment of future multicast-capable MAC protocols in terms of throughput efficiency it is necessary to know the maximum achievable throughput (capacity) of ring WDM networks for multicast traffic. In this paper we calculate the mean hop distances for multicast traffic in unidirectional and bidirectional ring WDM networks and the corresponding nominal capacities.

While the nominal capacity accounts for the mean hop distances travelled by the copies of a multicast packet on the different wavelength channels, it does not account for the relative location (alignment) of the paths taken by different packet copies. To understand the alignment of the paths, we need to take a closer look at the typically employed WDM ring network architecture. In a WDM ring network with destination stripping with the common fixed-tuned receiver node architecture, each network node is assigned one specific wavelength (home

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channel) for receiving packets, whereby a given wavelength may be the home channel for several nodes. Commonly, the nodes are assigned to the wavelengths in round robin fashion, as illustrated in Fig. 1. To reach a specific destination node, the sending node transmits the packet on the destination node's home channel. For instance, node 8 sends on wavelength 4 to reach node 4, as do all the other nodes. As a consequence the paths traversed by packets on a given wavelength tend to overlap. In our example, all the paths taken by the packets from the other nodes to node 4 have the ring segment from node 3 to node 4 in common. In other words, the ring segments on a given wavelength are unevenly loaded. Clearly, for a given fixed traffic pattern the achievable maximum long run average throughput of the network is limited by the ring segment with the highest load. We refer to this maximum throughput (stability limit) which is based on the maximum ring segment load as *effective capacity*. The effective capacity gives the maximum number of multicast packets (with a given traffic pattern) that can in the long run average be sent simultaneously. In this paper we analyze the maximum ring segment utilization probabilities for multicast traffic in unidirectional and bidirectional ring WDM networks and the corresponding effective capacities. This analysis builds on and incorporates the evaluation of the mean hop distances.

For the comprehensive assessment of future multicast capable MAC protocols for ring WDM networks it is important to know both nominal and effective capacities. The nominal and effective capacities evaluated in this paper can be used to assess the throughput efficiency of MAC protocols for ring WDM networks by considering the ratios of throughputs and capacities. In this paper we evaluate the nominal and effective capacities for three throughput metrics, namely the transmission, reception, and multicast throughput metrics which are defined shortly. These capacities enable the evaluation and comparison of future multicast MAC protocols for ring WDM networks in terms of transmitter, receiver, and multicast throughput efficiency.

The remainder of the paper is organized as follows. In Section I-A, we review related work. In Section II, we describe the model of the ring WDM network. In Section III, we formally define the capacities. We formally define the nominal capacities as well as the effective capacities for the transmitter, receiver, and multicast throughput metrics, i.e., we define a total of six capacities. In Sections IV and V, we analyze the average (mean) hop distances of both unidirectional and bidirectional ring WDM networks. The mean hop distances are the basis for evaluating the nominal capacities according to the definitions in Section III. In Sections VI and VII, we build on the analyzes of the mean hop distances and analyze the maximum ring segment utilization probabilities for unidirectional and bidirectional WDM ring networks. These utilization probabilities are the basis for evaluating the effective capacities according to the definitions in Section III. Numerical results are provided in Section VIII. Section IX concludes the paper.

A. Related Work

Multicasting in WDM ring networks has received relatively little attention to date [1], [9]. The photonics level issues involved in multicasting over ring WDM networks are explored in [10]. A node architecture suitable for multicasting in WDM ring networks is studied in [11]. The general network architecture and MAC protocol issues arising from multicasting in packet-switched WDM ring networks are addressed in [12], [13]. The fairness issues arising when transmitting a mix of unicast and multicast traffic in a ring WDM network are examined in [14]. These studies do not address the stability limit (capacity) of a packet-switched WDM ring network for multicast traffic, which is the main focus of this paper.

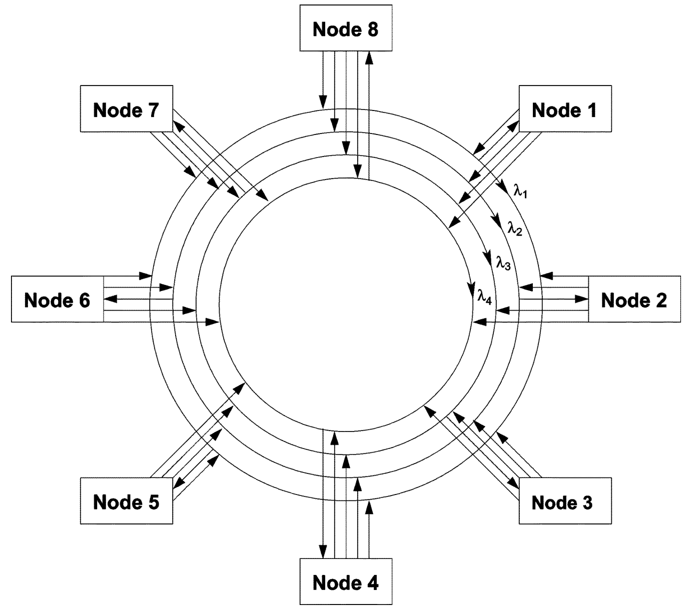

Fig. 1. Conceptual illustration of structure of unidirectional ring WDM network with $\Lambda = 4$ wavelength channels and $N = 8$ nodes with fixed-tuned receiver (FR) node architecture. To reach destination node(s) the sending node transmits the packet on the home channel(s) of the destination node(s).

We note for completeness that the wavelength assignment for multicasting in circuit-switched WDM ring networks, which are fundamentally different from the packet-switched networks considered in this paper, has been studied in [15]–[19]. We also note that for unicast traffic, the capacity jointly with the throughput efficiency of a packet-switched unidirectional ring WDM network have been examined in [20]. For unicast traffic, the throughputs achieved by different circuit-switched and packet-switched optical ring network architectures are compared in [21].

II. WDM Ring Network Model

We consider an optical wavelength division multiplexing (WDM) ring network which interconnects N network nodes. We initially consider a single-fiber ring network where the nodes are interconnected by one unidirectional fiber, as illustrated in Fig. 1, and subsequently a bidirectional dual-fiber ring network. In the case of the single-fiber ring network we number (index) the nodes sequentially as $n = 1, 2, \ldots, N$, in the transmission direction of the fiber, which we assume without loss of generality to be clockwise. In the bidirectional ring network we number the nodes sequentially in the clockwise direction. We suppose that each fiber carries Λ wavelength channels. In particular, we suppose that there is one set of wavelength channels $\lambda = \{1, \ldots, \Lambda\}$ in the single-fiber ring

TABLE I
OVERVIEW OF NOTATIONS

N	number of network nodes, indexed by $n = 1, 2, \ldots, N$
Λ	number of wavelengths in single-fiber ring, number of wavelengths in one direction in dual-fiber ring, indexed by $\lambda = 1, 2, \ldots, \Lambda$
η	number of nodes on a given home wavelength channel for reception, $\eta = N/\Lambda$
F	fanout (number of destinations) of a multicast packet, distributed according to $\mu_l = P(F = l),\ l = 1, 2, \ldots, N-1$
Δ	number of multicast packet copies transmitted for a multicast packet
$H_{s,\lambda}$	hop distance required on wavelength λ to serve multicast packet generated by node on wavelength s
$F_{s,\lambda}$	number of destination nodes reached by a multicast packet copy sent on wavelength λ by a node on wavelength s.
$\hat{C}_T$	nominal transmission capacity
$\hat{C}_R$	nominal reception capacity
$\hat{C}_M$	nominal multicast capacity, $\hat{C}_M = \hat{C}_T/E[\Delta] = \hat{C}_R/E[F]$
$u_{\max}$	largest segment utilization probability
C_M	effective multicast capacity, $C_M = 1/u_{\max}$
C_T	effective transmission capacity, $C_T = E[\Delta] \cdot C_M$
C_R	effective reception capacity, $C_R = E[F] \cdot C_M$

network and that there are two identical sets of the wavelength channels $\lambda = \{1, \ldots, \Lambda\}$ in the dual-fiber ring network, one set on the fiber in the clockwise direction, the other set on the fiber in the counterclockwise direction. Please refer to Table I for an overview of the notations. In the following discussion of the node structure and drop wavelength assignment we focus on the single-fiber network; for the dual-fiber network the node structure is duplicated for each fiber. We consider the family of node structures where each node 1) can transmit on any wavelength using either one or multiple tunable transmitters (TTs) or an array of Λ fixed-tuned transmitters (FTs), and 2) receive on one wavelength using a single fixed-tuned receiver (FR), which is a widely considered node structure [1], [20], [22]–[32].

For $N = \Lambda$ each node has its own separate *home channel* for reception. For $N > \Lambda$, each wavelength is shared by several nodes for the reception of packets. We let η denote the number of nodes that share a given wavelength as their home channel, i.e., $\eta = N/\Lambda$, and assume that η is an integer. In particular, the nodes $n = \lambda + k \cdot \Lambda$ with $k = 0, 1, \ldots, (\eta - 1)$ share the same drop wavelength (home channel) λ, $\lambda = 1, 2, \ldots, \Lambda$, i.e., have wavelength λ as their home channel. For brevity we will use the terminology that a node n is on wavelength λ if wavelength λ is the drop wavelength (home channel) of node n. A given node receiver terminates the wavelength channel on which it is homed. Nodes sharing the same wavelength may have to forward packets toward the destination node, resulting in *multihopping*. For unicast traffic, the destination node removes the packet from the ring. For multicast traffic, when a node receives a packet, it checks if there are additional destinations downstream; if so, it forward the packet to the other destinations; otherwise, the node is the last destination and removes the packet from the ring. With this destination release (stripping), wavelengths can be spatially reused by downstream nodes, leading to an increased network capacity.

We consider multicast traffic with a fanout (number of destination nodes) F that is described by the distribution

$$\mu_l := P(F = l), \quad l = 1, \ldots, N-1 \tag{1}$$

whereby $0 \leq \mu_l \leq 1$ and $\sum_{l=1}^{N-1} \mu_l = 1$. Note that unicast traffic is a special case of our multicast traffic model with $\mu_1 = 1$ and $\mu_l = 0$ for $l = 2, \ldots, N-1$. Similarly, our multicast traffic model also accommodates broadcast traffic for which $\mu_{N-1} = 1$ and $\mu_l = 0$ for $l = 1, \ldots, N-2$. Also note that throughout we assume that a multicast is not sent to the source node, hence the maximum fanout is $N-1$. As is common for capacity evaluations, we consider uniform traffic generation, i.e., all N nodes generate equivalent amounts of traffic. We consider uniform traffic destinations, i.e., the fanout set (set of destination nodes) $\mathcal{F}$ for a given multicast with given fanout F is drawn uniformly randomly from among the other $N-1$ nodes. Our analysis assumes that the source node, the fanout, and the set of destination nodes are drawn independently randomly. This independence assumption, however, is not critical for the analysis. Our results hold also for traffic patterns with correlations, as long as the long run average segment utilizations are equivalent to the utilizations with the independence assumption. For instance, our results hold for a correlated traffic model where a given source node transmits with a probability $p < 1$ to exactly the same set of destinations as the previous packet sent by the node, and with probability $1 - p$ to an independently randomly drawn number and set of destination nodes.

To transmit a multicast packet, the source node generates a copy of the multicast packet for each wavelength that is the drop wavelength for at least one of the destination nodes of the multicast. In other words, a multicast packet copy is generated for each wavelength that has at least one destination node of the multicast on it. We let Δ denote the number of multicast packet copies that are transmitted for a given multicast packet, i.e., the number of wavelengths with at least one destination node of the multicast. Note in particular that if all destination nodes are on the same (single) wavelength, then only one copy of the multicast packet is transmitted and if at least one destination node is on each of the Λ wavelengths, then $\Delta = \Lambda$ multicast copies are transmitted. We note that in the bidirectional ring network we do not allow for load balancing among both fibers, i.e., a given multicast transmission has to take place on that fiber which provides the smallest hop distance between the source node and the final multicast destination node.

III. Capacity Definitions

In this section we formally define the nominal transmission, reception, and multicast capacities as well as the effective transmission, reception, and multicast capacities. We proceed as follows with the definition of these capacities. First, we define the mean hop distance and the mean transmitter throughput. Based on these definitions we introduce the nominal transmission capacity. Next, we define the mean receiver and multicast throughputs, introduce the nominal reception and multicast capacities, and show how these can be derived from the nominal transmission capacity. Then we turn our attention to the effective capacity definitions. Starting from the definition of the multicast throughout we define the effective multicast capacity, and then show how to obtain the effective transmission and reception capacities from the effective multicast capacity.

A. Definition of Mean Hop Distance

In a source stripping network, where the source node takes a transmitted packet off the ring, each transmitted packet copy has to traverse the entire ring, which corresponds to a hop distance of N hops. With destination stripping, which we consider in this paper, the last destination node on the wavelength takes the multicast packet copy off the ring, allowing this node or nodes downstream to spatially reuse the wavelength and thus increasing the achievable throughput (capacity). More specifically, we define the hop distance that a given multicast packet copy travels on a given wavelength λ as the number of nodes that the packet copy visits, whereby each traversed node (irrespective of whether the node is on the wavelength λ or a different wavelength) as well as the last destination node on the wavelength counts as a visited node. Toward the definition of the mean hop distance we make a basic yet very important observation about the hop distance. We observe that the hop distance not only depends on the number of destination nodes that a packet copy needs to reach on a wavelength λ, but also on the relative alignment of the source node and the destination nodes on wavelength λ. (We note that this alignment issue is distinct from the relative alignment of the paths taken by the packet copies on *one* given wavelength, which gives rise to the uneven loading of the ring segments and the resulting need to consider the effective capacity.) For illustration suppose that node N, which is on wavelength Λ, is the source node of a broadcast that is destined to all other nodes in the network. Consider the packet copy sent on wavelength λ, $\lambda = 1, \dots, \Lambda - 1$, in the unidirectional ring network. In order to reach the first destination on the wavelength, the packet copy has to travel λ hops, and in order to reach all η destinations on the wavelength the packet copy has to travel $\lambda + (\eta - 1)\Lambda$ hops. (The situation is similar for $\lambda = \Lambda$, with the difference that there are only $\eta - 1$ destinations on wavelength Λ.) We observe that the initial "offset" and thus the total hop distance depends on the wavelength that the packet copy is sent on, or more generally on the relative alignment of the wavelength that the source node is on to the wavelengths that the destinations are on. To model this effect we let the random variable $H_{s,\lambda}$ with $s, \lambda = 1, \dots, \Lambda$, denote the hop distance required on wavelength λ to serve a multicast packet generated by a node on wavelength s.

For the further clarification of $H_{s,\lambda}$, let us consider the extreme case of unicast traffic. Suppose that a given source node on wavelength s sends a packet to a single destination node $d, d = 1, 2, \dots, N$, on wavelength $\lambda_d = d - \lfloor (d-1)/\Lambda \rfloor \cdot \Lambda$. In this case, the hop distance H_{s,λ_d} on the wavelength λ_d that the destination is on, is equal to the distance between the source node and the destination node in the transmission direction of the fiber in the unidirectional WDM ring. In the bidirectional WDM ring, H_{s,λ_d} equals the hop count between the source node and the destination node encountered on the fiber that provides the shortest path between the two nodes. There are no hop distances required on the other wavelengths $\lambda \neq \lambda_d$, i.e., $H_{s,\lambda} = 0$ for $\lambda = 1, \dots, \Lambda$, $\lambda \neq \lambda_d$. To model the mean hop distance of an actual packet copy transmission on a wavelength we consider the conditional expectation $E[H_{s,\lambda} | H_{s,\lambda} > 0]$. This conditional expectation gives the mean hop distance required on wavelength λ to serve a packet from a node on s given that the packet requires a copy transmission on λ. Note that

$$E[H_{s,\lambda}] = E[H_{s,\lambda} | H_{s,\lambda} > 0] \cdot P(H_{s,\lambda} > 0) + E[H_{s,\lambda} | H_{s,\lambda} = 0] \cdot P(H_{s,\lambda} = 0) \quad (2)$$

and that $E[H_{s,\lambda} | H_{s,\lambda} = 0] = 0$. Hence, this conditional expectation can be obtained as

$$E[H_{s,\lambda} | H_{s,\lambda} > 0] = \frac{E[H_{s,\lambda}]}{P(H_{s,\lambda} > 0)}. \quad (3)$$

We note that $P(H_{s,\lambda} > 0) = 1 - P(H_{s,\lambda} = 0)$, whereby $P(H_{s,\lambda} = 0)$ is the probability that a multicast packet generated by a node on wavelength s does not require a packet copy transmission on wavelength λ. We evaluate $P(H_{s,\lambda} = 0)$ by conditioning on the number of destination nodes of the multicast packet as

$$P(H_{s,\lambda} = 0) = \sum_{l=1}^{N-1} P(H_{s,\lambda} = 0 | F = l) \cdot \mu_l \quad (4)$$

$$= \begin{cases} \sum_{l=1}^{N-1} \frac{\binom{N-\eta}{l}}{\binom{N-1}{l}} \cdot \mu_l, & \text{if } s = \lambda \\ \sum_{l=1}^{N-1} \frac{\binom{N-\eta-1}{l}}{\binom{N-1}{l}} \cdot \mu_l, & \text{if } s \neq \lambda. \end{cases} \quad (5)$$

To see this first note that there are a total of $\binom{N-1}{l}$ ways of selecting the l destination nodes out of the $N-1$ eligible nodes (noting that the source node is not eligible to receive its own transmissions). If the packet copy is sent on the wavelength that the sender is on ($s = \lambda$), then there are $\binom{N-\eta}{l}$ ways of selecting the l destinations from among the $N - 1 - (\eta - 1)$ nodes on the other wavelengths. If the destination wavelength is different from the wavelength that the sender is on ($s \neq \lambda$), then there are $\binom{N-1-\eta}{l}$ ways of selecting the l destinations from among the $N - 1 - \eta$ nodes on the other wavelengths (i.e., such that there is no destination on the considered wavelength λ).

Note that $E[H_{s,\lambda} | H_{s,\lambda} > 0]$ gives the mean hop distance required on wavelength λ in order to serve a given multicast packet (which requires a copy transmission on λ) from a node on a particular wavelength s. In order to obtain the expected hop distance on a wavelength λ due to transmissions from all wavelengths $s, s = 1 \ldots, \Lambda$, we weigh the individual $E[H_{s,\lambda}|H_{s,\lambda} > 0]$ according to the contribution of transmissions from nodes on wavelength s toward the load on wavelength λ and define

$$E[H_\lambda|H_\lambda > 0] := \sum_{s=1}^{\Lambda} E[H_{s,\lambda}|H_{s,\lambda} > 0] \cdot \kappa \cdot P(H_{s,\lambda} > 0) \tag{6}$$

where κ is a normalization constant used to make $P(H_{s,\lambda} > 0)$, $s = 1, \ldots, \Lambda$, into a proper probability distribution with $\sum_{s=1}^{\Lambda} \kappa \cdot P(H_{s,\lambda} > 0) = 1$. That is,

$$\kappa = \frac{1}{\sum_{s=1}^{\Lambda} P(H_{s,\lambda} > 0)} \tag{7}$$

$$= \frac{1}{\sum_{s=1}^{\Lambda} [1 - P(H_{s,\lambda} = 0)]} \tag{8}$$

$$= \frac{1}{\Lambda - \sum_{l=1}^{N-1} \frac{\binom{N-\eta}{l}}{\binom{N-1}{l}} \cdot \mu_l - (\Lambda - 1) \cdot \sum_{l=1}^{N-1} \frac{\binom{N-\eta-1}{l}}{\binom{N-1}{l}} \cdot \mu_l}. \tag{9}$$

We thus obtain

$$E[H_\lambda|H_\lambda > 0] = \kappa \cdot \sum_{s=1}^{\Lambda} \frac{E[H_{s,\lambda}]}{P(H_{s,\lambda} > 0)} \cdot P(H_{s,\lambda} > 0) \tag{10}$$

$$= \kappa \sum_{s=1}^{\Lambda} E[H_{s,\lambda}]. \tag{11}$$

Note that by the rotational symmetry of the assignment of the drop wavelength to the network nodes (and of the traffic pattern), the expected value $E[H_\lambda|H_\lambda > 0]$ given in (11) is the same for all wavelengths $\lambda = 1, \ldots, \Lambda$. Formally we can express this rotational symmetry as

$$E[H_{s,\lambda}] = \begin{cases} E[H_{\Lambda,\lambda-s}], & \text{if } s < \lambda \\ E[H_{\Lambda,\Lambda-s+\lambda}], & \text{if } s \geq \lambda. \end{cases} \tag{12}$$

To see this note that sending a packet copy from a source node on wavelength s to destination nodes on a higher indexed wavelength λ, say from a source node on wavelength 1 to nodes on wavelength 2, is by the rotational symmetry equivalent to sending the packet copy from a source node on wavelength Λ to nodes on wavelength $\lambda - s$. Similarly, if the source node wavelength s is higher than the destination wavelength λ, then sending the packet is equivalent to sending it from a node on wavelength Λ to a node on wavelength $\Lambda - (s - \lambda)$. From these rotational symmetries it follows that

$$E[H_\lambda|H_\lambda > 0] = \kappa \sum_{\lambda=1}^{\Lambda} E[H_{\Lambda,\lambda}] \tag{13}$$

which is easily formally verified by inserting (12) in (11) and noting that each destination wavelength index occurs once in the resulting summation. Note that $E[H_\lambda|H_\lambda > 0]$ is independent of λ, which intuitively is due to the symmetric assignment of the drop wavelengths to the nodes (in conjunction with the uniform packet traffic pattern).

B. Definition of Mean Transmitter Throughput and Nominal Transmission Capacity

We define the mean transmitter throughput as the mean number of transmitters that are simultaneously transmitting multicast packet copies, whereby only the transmissions out of the source node are considered and not the transmissions by intermediate nodes that forward the multicast packet copy to additional destinations downstream.

We define the nominal transmission capacity $\hat{C}_T^\lambda$ on a single wavelength channel as

$$\hat{C}_T^\lambda := \frac{N}{E[H_\lambda|H_\lambda > 0]}. \tag{14}$$

Intuitively, $\hat{C}_T^\lambda$ is the constraint imposed by the required mean hop distance on the maximum mean number of simultaneously ongoing transmissions on a wavelength. Note that by our definition of the average hop distance on a wavelength and the rotational symmetry of the considered ring network, the transmission capacity $\hat{C}_T^\lambda$ is the same for all wavelengths $\lambda = 1, \ldots, \Lambda$. We define the nominal transmission capacity of the unidirectional single-fiber WDM ring network as

$$\hat{C}_T := \Lambda \cdot \hat{C}_T^\lambda \tag{15}$$

which is intuitively the maximum mean number of simultaneously ongoing transmissions in the ring network as limited by the mean hop distances required for the packet transmissions. For the bidirectional dual-fiber ring network with Λ wavelengths on each fiber we define the nominal transmission capacity analogously as $\hat{C}_T := 2 \cdot \Lambda \cdot \hat{C}_T^\lambda$.

C. Definition of Mean Receiver Throughput and Nominal Reception Capacity

We define the mean receiver throughput as the mean number of nodes that are receiving packets destined to them. We define the nominal reception capacity $\hat{C}_R$ as the mean receiver throughput corresponding to the nominal transmission capacity $\hat{C}_T$. The receiver throughput and the reception capacity depend critically on the number of destination nodes $F_{s,\lambda}$ reached by a given multicast packet copy sent on wavelength λ by a node on wavelength s. If each transmitted multicast packet copy has only one destination node on a wavelength, then the receiver throughput is equal to the transmitter throughput. On the other hand, if each multicast packet copy is destined to all η nodes on a wavelength, then the mean receiver throughput is η times the transmitter throughput.

Toward the evaluation of the mean receiver throughput we define the conditional expectation $E[F_{s,\lambda}|F_{s,\lambda} > 0]$ which is the mean number of receivers reached through the transmission on λ of the packet copy from a node on s given that the packet requires the transmission of a copy on λ. Clearly

$$E[F_{s,\lambda}|F_{s,\lambda} > 0] = \frac{E[F_{s,\lambda}]}{P(F_{s,\lambda} > 0)}. \tag{16}$$

Note that if a packet from a node on s has receivers on λ (i.e., if $F_{s,\lambda} > 0$), then the packet requires a hop count strictly larger than zero on λ, (i.e., $H_{s,\lambda} > 0$), i.e.,

$$P(F_{s,\lambda} > 0) = 1 - P(H_{s,\lambda} = 0). \tag{17}$$

Taking (17) into consideration and retracing the derivation of $E[H_\lambda | H_\lambda > 0]$ from (3) to (11) we obtain

$$E[F_\lambda | F_\lambda > 0] = \kappa \cdot \sum_{s=1}^{\Lambda} E[F_{s,\lambda}] \tag{18}$$

with κ given in (9). Noting that on average a fraction $(\eta - 1)/(N-1)$ of the expected total number of destination nodes $E[F] = \sum_{l=1}^{N-1} l \cdot \mu_l$ is on the wavelength s of the sender and that on average a fraction $\eta/(N-1)$ of $E[F]$ is on each of the other wavelength, i.e.

$$E[F_{s,\lambda}] = \begin{cases} \frac{\eta-1}{N-1} \cdot E[F], & \text{if } s = \lambda \\ \frac{\eta}{N-1} \cdot E[F], & \text{if } s \neq \lambda \end{cases} \tag{19}$$

we obtain

$$E[F_\lambda | F_\lambda > 0] = \kappa \cdot E[F] = \kappa \cdot \sum_{l=1}^{N-1} l \cdot \mu_l. \tag{20}$$

We define the nominal reception capacity of a single wavelength channel as

$$\hat{C}_R^\lambda := E[F_\lambda | F_\lambda > 0] \cdot \hat{C}_T^\lambda \tag{21}$$

and define the reception capacity of the unidirectional single-fiber WDM ring network as

$$\hat{C}_R := \Lambda \cdot \hat{C}_R^\lambda \tag{22}$$

which is intuitively the maximum mean number of receivers in the ring network that can simultaneously receive packets destined to them when considering only the mean hop distance as limiting the throughput. For the bidirectional dual-fiber ring network with Λ wavelengths on each fiber we define the nominal reception capacity analogously as $\hat{C}_R := 2 \cdot \Lambda \cdot \hat{C}_R^\lambda$.

D. Definition of Mean Multicast Throughput and Nominal Multicast Capacity

Following [33] we define the mean multicast throughput as the mean number of multicasts that are simultaneously transmitted in the network. We note that in general the mean multicast throughput is obtained by dividing the mean transmitter throughput by the mean number of packet copy transmissions $E[\Delta]$ required to serve a multicast. We first derive $E[\Delta]$ by noting that Δ is equivalent to the number of wavelength channels with at least one destination node of the multicast. The mean of Δ is given by

$$E[\Delta] = \sum_{\lambda=1}^{\Lambda} E[1_{\{H_\lambda \geq 1\}}] = \Lambda - \sum_{\lambda=1}^{\Lambda} E[1_{\{H_\lambda = 0\}}] \tag{23}$$

$$= \Lambda - \sum_{\lambda=1}^{\Lambda} P(H_\lambda = 0) \tag{24}$$

$$= \Lambda - \sum_{\lambda=1}^{\Lambda} \frac{1}{\Lambda} \sum_{s=1}^{\Lambda} P(H_{s,\lambda} = 0) \tag{25}$$

where (25) follows by noting that with the considered uniform traffic generation, the source node is equally likely on any of the Λ wavelengths. With $P(H_{s,\lambda} = 0)$ from (5) we obtain

$$E[\Delta] = \Lambda - \sum_{l=1}^{N-1} \mu_l \left\{ \frac{\binom{N-1-\eta}{l}}{\binom{N-1}{l}} (\Lambda - 1) + \frac{\binom{N-\eta}{l}}{\binom{N-1}{l}} \right\}. \tag{26}$$

We define the nominal multicast capacity as the maximum mean number of multicasts that can be simultaneously transmitted in the network subject to the mean hop distances traveled by the packet copies, i.e.

$$\hat{C}_M := \frac{\hat{C}_T}{E[\Delta]}. \tag{27}$$

E. Relationships Between Transmission, Reception, and Multicast Capacities

We note that the nominal multicast capacity can equivalently be expressed in terms of the nominal reception capacity or the mean hop distance. In particular

$$\hat{C}_M = \frac{\hat{C}_T}{E[\Delta]} = \frac{\hat{C}_R}{E[F]} = \begin{cases} \frac{N \cdot \Lambda}{\Lambda \cdot E[H_\lambda]}, & \text{for unidir. ring netw.} \\ \frac{2 \cdot N \cdot \Lambda}{\Lambda \cdot E[H_\lambda]}, & \text{for bidir. ring netw.} \end{cases} \tag{28}$$

which can be verified from the respective definitions of these metrics as follows. We start from the last expression in (28), namely $\hat{C}_M = N\Lambda/(\Lambda E[H_\lambda])$ for the case of the unidirectional ring (the case of the bidirectional ring is analogous), and show that it is identical to the first two expressions. We proceed as follows:

$$\frac{N}{E[H_\lambda]} = \frac{N}{E[H_\lambda | H_\lambda > 0] \cdot P(H_\lambda > 0)} \tag{29}$$

$$= \frac{N}{\frac{N}{\hat{C}_T^\lambda} \cdot P(H_\lambda > 0)} = \frac{\hat{C}_T}{\Lambda \cdot P(H_\lambda > 0)}$$

$$= \frac{\hat{C}_T}{\Lambda \cdot P(F_\lambda > 0)} \tag{30}$$

$$= \frac{\hat{C}_T}{E[\Delta]}. \tag{31}$$

Next

$$\hat{C}_M = \frac{\hat{C}_T}{E[\Delta]} = \frac{\hat{C}_R}{E[F_\lambda | F_\lambda > 0] \cdot E[\Delta]} \tag{32}$$

$$= \frac{\hat{C}_R}{E[F_\lambda | F_\lambda > 0] \cdot P(F_\lambda > 0) \cdot \Lambda} = \frac{\hat{C}_R}{E[F_\lambda] \cdot \Lambda} \tag{33}$$

$$= \frac{\hat{C}_R}{E[F]}. \tag{34}$$

Note that the last expression in (28) embodies the following intuitive notion of the nominal multicast capacity. The unidirectional ring network provides a resource of $N \cdot \Lambda$ ring segments (hops); $2 \cdot \Lambda \cdot N$ in the case of the bidirectional ring network. A given packet requires on average a total hop distance of $\Lambda \cdot E[H_\lambda]$ for service in the network. Thus, the ratio $N \cdot \Lambda/(\Lambda \cdot E[H_\lambda])$ gives the number of packets that "fit" simultaneously on the wavelength channels of the ring network, or equivalently the maximum mean number of packets that can simultaneously be served in the long run in the network.

F. Definition of Effective Multicast Capacity

We define the effective multicast capacity C_M as the maximum mean number of multicasts (for the considered fixed traffic pattern) that can be transmitted simultaneously in the network. The effective multicast capacity is limited by the ring segments with the highest utilization. We proceed to formally define the effective multicast capacity for the unidirectional ring network and then for the bidirectional ring network.

Toward the formal definition of C_M for the unidirectional ring we introduce the following definitions. We define the *ring segment* s, $s = 1, \ldots, N$, as the segment of the fiber ring between node $s-1$ and node s (in case of $s = 1$, between nodes N and 1). We let $u(\lambda, s)$ denote the probability that a given arbitrary multicast from a given arbitrary sending node traverses (utilizes) the ring segment s on wavelength λ. We let $u_{\max}$ denote the largest utilization probability in the network, i.e.

$$u_{\max} = \max_{s,\lambda}\{u(\lambda, s)\}. \tag{35}$$

With these definitions, the effective multicast capacity is given by

$$C_M = \frac{1}{u_{\max}}. \tag{36}$$

For the considered unidirectional ring network, the maximum utilization probability $u_{\max}$ is attained on every wavelength λ, $\lambda = 1, \ldots, \Lambda$, for every segment s that corresponds to a node on wavelength λ, i.e., for segments $s = \lambda + k \cdot \Lambda, k = 0, \ldots, \eta - 1$, on wavelength λ.

For the definition of the effective multicast capacity in the bidirectional ring network we consider the maximum utilization probability of the ring segments similar to the case of the unidirectional ring. One key distinction of the bidirectional ring is that we need to consider the *direction* of the ring segments. We let $u(\lambda, +, s)$ denote the probability that a given arbitrary multicast from a given arbitrary sending node traverses the ring segment s on wavelength λ in the clockwise direction, and denote $u(\lambda, -, s)$ for the corresponding probability for the counterclockwise direction. We let $u_{\max}$ denote the largest utilization probability, i.e.

$$u_{\max} = \max_{s,\lambda}\{u(\lambda, +, s),\ u(\lambda, -, s)\}. \tag{37}$$

The effective multicast capacity is then given by the reciprocal of $u_{\max}$ as in (36).

To intuitively illustrate the meaning of the effective multicast capacity, consider a scenario where each of the N network nodes generates multicast packets of mean size L (bits) at a long run average rate of σ (packets/second) and suppose the transmission rate on each wavelength channel is R (bit/s). Then the network is stable if

$$N \cdot \sigma < \frac{R}{L} \cdot C_M. \tag{38}$$

G. Definition of Effective Transmission and Reception Capacities

We define the effective transmission capacity C_T and the effective reception capacity C_R as the maximum mean transmitter and receiver throughputs for the given traffic pattern. These capacities can be calculated from C_M using the relationships in (28). In particular

$$C_T = E[\Delta] \cdot C_M \text{ and } C_R = E[F] \cdot C_M. \tag{39}$$

IV. Analysis of Mean Hop Distance in Unidirectional Ring Network

In this section we evaluate the mean hop distance $E[H_{\Lambda,\lambda}]$ for $\lambda = 1, \ldots, \Lambda$ for the unidirectional ring network, where the network nodes are numbered sequentially in the clockwise direction and packet copies are sent in the clockwise direction around the ring. The derived $E[H_{\Lambda,\lambda}]$ can be inserted in (13) to obtain the mean hop distance $E[H_\lambda | H_\lambda > 0]$ on a wavelength, which in turn is used in the calculation of the nominal capacities as defined in Sections III-B–D. Without loss of generality we suppose that the source node of the multicast is node N, which is on wavelength Λ. (This choice of the source node index simplifies the notation in the analysis in that node λ is exactly λ hops from the source node; however our analysis is general applying to any source node.) Note that the fanout $F_{\Lambda,\lambda}$ (number of destination nodes) of the considered multicast on wavelength λ, satisfies $0 \le F_{\Lambda,\lambda} \le \eta - 1$ for $\lambda = \Lambda$ (since the source node does not send to itself) and $0 \le F_{\Lambda,\lambda} \le \eta$ for $\lambda = 1, \ldots, \Lambda - 1$. With $\mathcal{F}_{\Lambda,\lambda}$ denoting the fanout set (set of destination nodes) of the considered multicast on wavelength λ we define the hop distance required on wavelength λ to serve the multicast packet from node N on wavelength Λ formally as

$$H_{\Lambda,\lambda} = \max\{n : n \in \mathcal{F}_{\Lambda,\lambda}\}. \tag{40}$$

Note that $H_{\Lambda,\Lambda}$ can take on the values $k \cdot \Lambda$ for $k = 0, 1, \ldots, (\eta - 1)$. The value 0 indicates that the given multicast packet does not have a destination on wavelength Λ. Similarly, note that $H_{\Lambda,\lambda}$, $\lambda = 1, \ldots, \Lambda - 1$, can take on the values 0 and $\lambda + k \cdot \Lambda$ for $k = 0, \ldots, (\eta - 1)$. Note that there are a total of η possible values for $H_{\Lambda,\Lambda}$ and a total of $\eta + 1$ possible values for $H_{\Lambda,\lambda}$, $\lambda = 1, \ldots, \Lambda - 1$. We now proceed to evaluate the probabilities $P(H_{\Lambda,\lambda} \le n)$, which we use in turn to evaluate $E[H_{\Lambda,\lambda}]$.

A. Evaluation of $P(H_{\Lambda,\Lambda} \le n)$ and $E[H_{\Lambda,\Lambda}]$

We first consider the wavelength $\lambda = \Lambda$. We begin by conditioning the probability $P(H_{\Lambda,\Lambda} \le k\Lambda)$ on the number $F_{\Lambda,\Lambda}$ of multicast destinations on wavelength Λ, i.e.,

$$P(H_{\Lambda,\Lambda} \le k\Lambda) = \sum_{i=0}^{k} P(H_{\Lambda,\Lambda} \le k\Lambda | F_{\Lambda,\Lambda} = i) \cdot P(F_{\Lambda,\Lambda} = i) \tag{41}$$

$$= \sum_{i=0}^{k} P(H_{\Lambda,\Lambda} \le k\Lambda | F_{\Lambda,\Lambda} = i) \cdot \sum_{l=1}^{N-1} P(F_{\Lambda,\Lambda} = i | F = l) \cdot \mu_l. \quad (42)$$

For the calculation of $P(H_{\Lambda,\Lambda} \le k\Lambda | F_{\Lambda,\Lambda} = i)$ we note that there are a total of $\binom{\eta-1}{i}$ ways of selecting the i destination nodes among the $\eta - 1$ (eligible) nodes on the considered wavelength Λ, and there are $\binom{k}{i}$ ways of selecting the i destination nodes from among the nodes with indices less than or equal to $k\Lambda$ on wavelength Λ. Hence

$$P(H_{\Lambda,\Lambda} \le k\Lambda | F_{\Lambda,\Lambda} = i) = \frac{\binom{k}{i}}{\binom{\eta-1}{i}}. \quad (43)$$

For the calculation of the conditional probability $P(F_{\Lambda,\Lambda} = i | F = l)$ note that there are $\binom{N-1}{l}$ ways of selecting l destination nodes out of the possible $N-1$ destination nodes. At the same time, there are $\binom{\eta-1}{i} \cdot \binom{N-\eta}{l-i}$ ways of selecting i destination nodes out of the $\eta - 1$ (eligible) nodes on the considered wavelength Λ and $l - i$ destination nodes out of the $N - 1 - (\eta - 1)$ nodes on the other wavelengths. Hence

$$P(F_{\Lambda,\Lambda} = i | F = l) = \frac{\binom{\eta-1}{i}\binom{N-\eta}{l-i}}{\binom{N-1}{l}}. \quad (44)$$

Inserting (43) and (44) into (42) we obtain

$$P(H_{\Lambda,\Lambda} \le k\Lambda) = \sum_{i=0}^{k} \binom{k}{i} \sum_{l=1}^{N-1} \frac{\binom{N-\eta}{l-i}}{\binom{N-1}{l}} \cdot \mu_l. \quad (45)$$

Next, we evaluate the expected value of the hop distance that a copy of the given multicast travels on the considered wavelength Λ. In other words we calculate the expected value of the maximum index of a multicast destination on wavelength Λ. We proceed as follows:

$$E[H_{\Lambda,\Lambda}] = \Lambda \sum_{k=0}^{\eta-2} P(H_{\Lambda,\Lambda} > k\Lambda) \quad (46)$$

$$= \Lambda \sum_{k=0}^{\eta-2} (1 - P(H_{\Lambda,\Lambda} \le k\Lambda)) \quad (47)$$

$$= \Lambda(\eta - 1) - \Lambda \sum_{k=0}^{\eta-2} \sum_{i=0}^{k} \binom{k}{i} \cdot \sum_{l=1}^{N-1} \frac{\binom{N-\eta}{l-i}}{\binom{N-1}{l}} \mu_l \quad (48)$$

$$= \Lambda(\eta - 1) - \Lambda \sum_{l=1}^{N-1} \sum_{k=0}^{\eta-2} \frac{1}{\binom{N-1}{l}} \cdot \sum_{i=0}^{k} \binom{k}{i} \binom{N-\eta}{l-i} \mu_l \quad (49)$$

$$= \Lambda(\eta - 1) - \Lambda \sum_{l=1}^{N-1} \sum_{k=0}^{\eta-2} \frac{\binom{N+k-\eta}{l}}{\binom{N-1}{l}} \mu_l \quad (50)$$

$$= \Lambda(\eta - 1) - \Lambda \sum_{l=1}^{N-1} \frac{\binom{N-1}{l+1} - \binom{N-\eta}{l+1}}{\binom{N-1}{l}} \mu_l \quad (51)$$

$$= N - \Lambda N \sum_{l=1}^{N-1} \frac{\mu_l}{l+1} + \Lambda \sum_{l=1}^{N-1} \frac{\binom{N-\eta}{l+1}}{\binom{N-1}{l}} \mu_l \quad (52)$$

where (50) follows from the properties of the hypergeometric distribution and (51) is obtained by noting that $\sum_{k=0}^{m} \binom{k}{i} = \binom{m+1}{i+1}$.

B. Evaluation of $P(H_{\Lambda,\lambda} \le n)$ and $E[H_{\Lambda,\lambda}]$ for $\lambda = 1, \ldots, \Lambda - 1$

The calculation of $P(H_{\Lambda,\lambda} \le n)$ for $\lambda = 1, \ldots, \Lambda - 1$ proceeds analogously to the calculation above by conditioning on the number of destination nodes $F_{\Lambda,\lambda}$ on wavelength λ. We first note that $P(H_{\Lambda,\lambda} = 0 | F_{\Lambda,\lambda} = 0) = 1$ and that $P(H_{\Lambda,\lambda} = 0 | F_{\Lambda,\lambda} = i) = 0$ for $i \ge 1$. The calculation of $P(H_{\Lambda,\lambda} \le \lambda + k\Lambda)$ for $k = 0, \ldots, \eta - 1$ is analogous to the calculations above with the main difference that there are now a total of $\binom{\eta}{i}$ ways of selecting the i destination nodes among the now η (eligible) nodes on the considered wavelength. Also, there are now $\binom{k+1}{i}$ ways of selecting the i destination nodes from among the nodes with indices less than or equal to $\lambda + k\Lambda$. Thus

$$P(H_{\Lambda,\lambda} \le \lambda + k\Lambda | F_{\Lambda,\lambda} = i) = \frac{\binom{k+1}{i}}{\binom{\eta}{i}}. \quad (53)$$

Furthermore, noting that there are now $\binom{\eta}{i} \cdot \binom{N-\eta-1}{l-i}$ ways of selecting i destination nodes out of the η (eligible) nodes on the considered wavelength λ and $l - i$ destination nodes out of the $N - 1 - \eta$ nodes on the other wavelengths we obtain

$$P(F_{\Lambda,\lambda} = i | F = l) = \frac{\binom{\eta}{i}\binom{N-\eta-1}{l-i}}{\binom{N-1}{l}}. \quad (54)$$

For the calculation of the mean hop distance we proceed as above, paying attention to the fact that if there is a multicast destination on wavelength λ it is at least λ hops away from the considered source node N. Thus

$$E[H_{\Lambda,\lambda}] = \sum_{n=0}^{N} P(H_{\Lambda,\lambda} > n) \quad (55)$$

$$= \lambda[1 - P(H_{\Lambda,\lambda} = 0)] + \Lambda \sum_{k=0}^{\eta-2} P(H_{\Lambda,\lambda} > \lambda + k\Lambda) \quad (56)$$

$$= \lambda[1 - P(F_{\Lambda,\lambda} = 0)] + \Lambda \sum_{k=0}^{\eta-2} (1 - P(H_{\Lambda,\lambda} \le \lambda + k\Lambda)) \quad (57)$$

$$= \lambda[1 - P(F_{\Lambda,\lambda} = 0)] + \Lambda(\eta - 1) - \Lambda \sum_{k=0}^{\eta-2} \sum_{i=0}^{k+1} P(H_{\Lambda,\lambda} \leq \lambda + k\Lambda | F_{\Lambda,\lambda} = i) \cdot P(F_{\Lambda,\lambda} = i) \quad (58)$$

$$= \lambda[1 - P(F_{\Lambda,\lambda} = 0)] + \Lambda(\eta - 1) - \Lambda(\eta - 1) \cdot P(F_{\Lambda,\lambda} = 0) - \Lambda \sum_{k=0}^{\eta-2} \sum_{i=1}^{k+1} P(H_{\Lambda,\lambda} \leq \lambda + k\Lambda | F_{\Lambda,\lambda} = i) \cdot P(F_{\Lambda,\lambda} = i). \quad (59)$$

Inserting (53) and (54) and simplifying as detailed in Appendix A we obtain

$$E[H_{\Lambda,\lambda}] = \lambda \left(1 - \sum_{l=1}^{N-1} \frac{\binom{N-\eta-1}{l}}{\binom{N-1}{l}} \mu_l \right) + N - \Lambda N \sum_{l=1}^{N-1} \frac{\mu_l}{l+1} + \Lambda \sum_{l=1}^{N-1} \frac{\binom{N-\eta}{l+1}}{\binom{N-1}{l}} \mu_l. \quad (60)$$

Note that the last three terms of (60) are identical to (52), which is bounded by $N - \Lambda$. The first term in (60) accounts for the "offset" required to reach the first node on wavelength λ from node N on wavelength Λ.

For the special case of unicast traffic with $\mu_1 = 1$ and $\mu_l = 0$ for $l = 2, \ldots, N-1$ we obtain with the mean hop distances $E[H_{\Lambda,\lambda}]$, $\lambda = 1, \ldots, \Lambda$, derived in this section and the preceding section from (13) the conditional mean hop distance $E[H_\lambda | H_\lambda > 0] = N/2$, and the nominal capacities are hence $\hat{C}_T = \hat{C}_R = \hat{C}_M = \Lambda \cdot N/(N/2) = 2 \cdot \Lambda$, i.e., exactly twice the number of wavelength channels, for unicast traffic in the unidirectional ring WDM network.

V. Analysis of Mean Hop Distance in Bidirectional Ring Network

In this section we evaluate the mean hop distance $E[H_{\Lambda,\lambda}]$, $\lambda = 1, \ldots, \Lambda$, required on wavelength λ in the bidirectional WDM ring with shortest path steering in order to serve a multicast packet generated by a node on wavelength Λ, which we assume without loss of generality to be node N. In the bidirectional ring network, the packet copy is sent in the direction—either clockwise or counterclockwise—that results in the smaller hop distance. We let $H^+_{\Lambda,\lambda}$ denote the hop distance required to reach all destinations of the multicast on wavelength λ on the fiber running in the clockwise direction and note that

$$H^+_{\Lambda,\lambda} = \max\{m : m \in \mathcal{F}_{\Lambda,\lambda}\}. \quad (61)$$

To see this, note that when node N sends a multicast packet copy in the clockwise direction (in which the nodes are sequentially indexed) then the number of visited nodes is equal to the index of the node with the highest index among the destination nodes on the considered wavelength λ. Similarly, we let $H^-_{\Lambda,\lambda}$ denote the hop distance required to reach all destinations of the multicast on wavelength λ on the fiber running in the counter clockwise direction and note that

$$H^-_{\Lambda,\lambda} = N - \min\{m : m \in \mathcal{F}_{\Lambda,\lambda}\}. \quad (62)$$

This is because the destination node with the smallest index governs how far the packet copy needs to travel in the counter clockwise direction.

Since the multicast copy is sent in the direction with the smaller hop count, the actual hop distance $H_{\Lambda,\lambda}$ on wavelength λ is given as the minimum of the hop distances in the clockwise and counter clockwise directions, i.e.,

$$H_{\Lambda,\lambda} = \min\left\{H^+_{\Lambda,\lambda},\ H^-_{\Lambda,\lambda}\right\} \quad (63)$$

whereby we define $H_{\Lambda,\lambda} = 0$ if $\mathcal{F}_{\Lambda,\lambda} = \emptyset$. The goal is now to calculate the expected value of the hop distance, i.e., to calculate $E[H_{\Lambda,\lambda}]$, for $\lambda = 1, \ldots, \Lambda$. We first consider the wavelengths $\lambda = 1, \ldots, \Lambda - 1$, and consider the wavelength $\lambda = \Lambda$, on which the considered source node N is, later.

A. Evaluation of $E[H_{\Lambda,\lambda}]$ for $\lambda = 1, \ldots, \Lambda - 1$

We note that by symmetry

$$E[H_{\Lambda,\lambda}] = E[H_{\Lambda,\Lambda-\lambda}]. \quad (64)$$

Hence it suffices to consider $\lambda = 1, \ldots, \lfloor \Lambda/2 \rfloor$. We define

$$p_{\lambda,m,+,i} := P(H_{\Lambda,\lambda} = H^+_{\Lambda,\lambda} = m | F_{\Lambda,\lambda} = i) \quad (65)$$

$$p_{\lambda,m,-,i} := P\left(H^+_{\Lambda,\lambda} > H^-_{\Lambda,\lambda},\ H^-_{\Lambda,\lambda} = N - m | F_{\Lambda,\lambda} = i\right) \quad (66)$$

for $i = 1, \ldots, \eta$, and $m = \lambda, \lambda + \Lambda, \ldots, \lambda + (\eta - 1) \cdot \Lambda$. Intuitively, $p_{\lambda,m,+,i}$ is the probability that given there are i multicast destinations on wavelength λ, the highest indexed multicast destination is node m and the smaller hop count is achieved by sending the packet copy in the clockwise direction. Analogously, $p_{\lambda,m,-,i}$ considers the case where m is the lowest indexed multicast destination node and the smaller hop count is achieved by sending the packet copy in the counterclockwise direction. We evaluate the expected hop distance as

$$E[H_{\Lambda,\lambda}] = \sum_{i=1}^{\eta} E[H_{\Lambda,\lambda} | F_{\Lambda,\lambda} = i] \cdot P(F_{\Lambda,\lambda} = i) \quad (67)$$

$$= \sum_{i=1}^{\eta} E[H_{\Lambda,\lambda} | F_{\Lambda,\lambda} = i] \sum_{l=1}^{N-1} P(F_{\Lambda,\lambda} = i | F = l) \mu_l \quad (68)$$

$$= \sum_{i=1}^{\eta} E[H_{\Lambda,\lambda} | F_{\Lambda,\lambda} = i] \cdot \sum_{l=1}^{N-1} \frac{\binom{\eta}{i} \binom{N-1-\eta}{l-i}}{\binom{N-1}{l}} \mu_l. \quad (69)$$

For the binomial expression in (69) note that there are $\binom{\eta}{i} \binom{N-1-\eta}{l-i}$ ways of selecting i multicast destinations on the considered wavelength λ and the remaining $l - i$ destinations on the other wavelengths. Also, there are a total of $\binom{N-1}{l}$ ways of selecting l destinations from among the $N - 1$ possible nodes. Furthermore, note that

$$E[H_{\Lambda,\lambda} | F_{\Lambda,\lambda} = i] = \sum_{k=0}^{\eta-1} \{p_{\lambda,\lambda+k\cdot\Lambda,+,i} \cdot (\lambda + k \cdot \Lambda) + p_{\lambda,\lambda+k\cdot\Lambda,-,i} \cdot (N - (\lambda + k \cdot \Lambda))\}. \quad (70)$$

It thus remains to calculate $p_{\lambda,m,+,i}$ and $p_{\lambda,m,-,i}$. For the calculation of these probabilities we distinguish two cases: The case that $\lambda + k\cdot\Lambda \le N/2$, i.e., that the node with highest index $\lambda + k\cdot\Lambda$ lies at most half way around the ring in the clockwise direction from the source node N, and the complementary case $\lambda + k\cdot\Lambda > N/2$. For the case $\lambda + k\cdot\Lambda \le N/2$, we obtain

$$p_{\lambda,\lambda+k\cdot\Lambda,+,i} = P\{\lambda + k\cdot\Lambda \in \mathcal{F}_{\Lambda,\lambda}, \lambda + r \cdot\Lambda \notin \mathcal{F}_{\Lambda,\lambda}\ \forall r > k | F_{\Lambda,\lambda} = i\} \quad (71)$$

$$= P\{\lambda + k\cdot\Lambda \in \mathcal{F}_{\Lambda,\lambda} | F_{\Lambda,\lambda} = i\} \cdot P\{\lambda + r\cdot\Lambda \notin \mathcal{F}_{\Lambda,\lambda}\ \forall r > k | F_{\Lambda,\lambda} = i, \lambda + k\cdot\Lambda \in \mathcal{F}_{\Lambda,\lambda}\} \quad (72)$$

$$= \frac{i}{\eta}\cdot\frac{\binom{k}{i-1}}{\binom{\eta-1}{i-1}}. \quad (73)$$

To see this, note that with probability i/η falls a multicast destination on the considered node $\lambda + k\cdot\Lambda$, i.e., the kth node on the considered wavelength. With k fixed, there are $\binom{k}{i-1}$ ways of selecting the remaining $i-1$ destinations from among the nodes $\lambda, \lambda+\Lambda, \ldots, \lambda+(k-1)\cdot\Lambda$, i.e., the k nodes with lower indices on the considered wavelength λ. At the same time, there are a total of $\binom{\eta-1}{i-1}$ ways of selecting the remaining $i-1$ destinations from among the $\eta-1$ other nodes on the wavelength. We also obtain

$$p_{\lambda,\lambda+k\cdot\Lambda,-,i} = P\{\lambda + k\cdot\Lambda \in \mathcal{F}_{\Lambda,\lambda}, \lambda + r\cdot\Lambda \notin \mathcal{F}_{\Lambda,\lambda}\ \forall\, r < k, m \in \mathcal{F}_{\Lambda,\lambda} \text{ for some } m > N - (\lambda + k\cdot\Lambda) | F_{\Lambda,\lambda} = i\} \quad (74)$$

$$= \frac{i}{\eta}\frac{\binom{\eta-k-1}{i-1}}{\binom{\eta-1}{i-1}}\left(1 - \frac{\binom{\eta-2\cdot k-1}{i-1}}{\binom{\eta-k-1}{i-1}}\right). \quad (75)$$

To follow (74) note that if the node $\lambda + k\cdot\Lambda$ $(\le N/2)$ is the lowest indexed node, it is preferable to send the packet copy counter clockwise only if there is at least one node with index higher than $N - (\lambda + k\cdot\Lambda)$. We obtain then (75) with the same reasoning as above about the selection of the considered node $\lambda + k\cdot\Lambda$ and noting that there are $\binom{\eta-k-1}{i-1}$ ways of selecting the remaining $i-1$ nodes from among the nodes $\lambda + (k+1)\cdot\Lambda, \ldots, \lambda+(\eta-1)\cdot\Lambda$, i.e., the $\eta-1-k$ nodes with higher indices on the considered wavelength λ. We furthermore note that for the (due to the symmetry $E[H_{\Lambda,\lambda}] = E[H_{\Lambda,\Lambda-\lambda}]$) considered cases of $\lambda \le \Lambda/2$, there are exactly k nodes on wavelength λ with indices higher than $N-(\lambda+k\cdot\Lambda)$. Hence the event that at least one destination node is selected from among these k nodes is complementary to the event that the remaining $i-1$ nodes are selected from among the $\eta-1-k-k$ nodes with indices higher than $\lambda + k\cdot\Lambda$ and less than or equal to $N - (\lambda + k\cdot\Lambda)$ on wavelength λ. The case $\lambda + k\cdot\Lambda > N/2$ is analyzed with reasoning similar to above, as

$$p_{\lambda,\lambda+k\cdot\Lambda,+,i} = P\{\lambda + k\cdot\Lambda \in \mathcal{F}_{\Lambda,\lambda}, \lambda + r\cdot\Lambda \notin \mathcal{F}_{\Lambda,\lambda}\ \forall\, r > k, m \in \mathcal{F}_{\Lambda,\lambda} \text{ for some } m \le N - (\lambda + k\cdot\Lambda) | F_{\Lambda,\lambda} = i\} \quad (76)$$

$$= \frac{i}{\eta}\cdot\frac{\binom{k}{i-1}}{\binom{\eta-1}{i-1}}\left(1 - \frac{\binom{2k-\eta}{i-1}}{\binom{k}{i-1}}\right). \quad (77)$$

To follow (76) note that if the node $\lambda + k\cdot\Lambda$ $(> N/2)$ is the highest indexed node, the packet copy is sent in the clockwise direction only if there is at least one node with index lower than or equal to $N-(\lambda+k\cdot\Lambda)$, of which there are $\eta-k$. To see this, note that there are k nodes with indices less than $\lambda + k\cdot\Lambda$, and $\eta-k$ nodes with indices higher than or equal to $\lambda+k\cdot\Lambda$. Now, the event that there is at least one destination with index lower than $\lambda + k\cdot\Lambda$ is complementary to the event that the remaining $i-1$ destination nodes are selected from among the nodes with indices higher than $N - (\lambda + k\cdot\Lambda)$ and lower than k, of which there are $k - (\eta - k)$.

We also obtain

$$p_{\lambda,\lambda+k\cdot\Lambda,-,i} = P\{\lambda + k\cdot\Lambda \in \mathcal{F}_{\Lambda,\lambda}, \lambda + r\cdot\Lambda \notin \mathcal{F}_{\Lambda,\lambda}\ \forall\, r < k | F_{\Lambda,\lambda} = i\} \quad (78)$$

$$= \frac{i}{\eta}\cdot\frac{\binom{\eta-k-1}{i-1}}{\binom{\eta-1}{i-1}}. \quad (79)$$

Inserting the obtained expressions for $p_{\lambda,\lambda+k\cdot\Lambda,+,i}$ and $p_{\lambda,\lambda+k\cdot\Lambda,-,i}$ in (70) and (67) we obtain after some algebraic manipulations which are detailed in Appendix B

$$E[H_{\Lambda,\lambda}] = 2N + \Lambda - 2N\Lambda\sum_{l=1}^{N-1}\frac{\mu_l}{l+1} - \lambda\sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}}\sum_{j=1}^{\eta}(-1)^j\binom{N-1-j}{l-1} + \Lambda\sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}} \cdot\left\{\binom{N-1-\eta}{l} + 2\binom{N-1-\eta}{l+1} - \sum_{k=0}^{\lfloor\frac{\eta-1}{2}\rfloor}(\eta-k)\binom{N-2k-2}{l-1} - \sum_{k=\lfloor\frac{\eta-1}{2}\rfloor+1}^{\eta-1} k\binom{N+2k-2\eta-1}{l-1}\right\}. \quad (80)$$

B. Evaluation of $E[H_{\Lambda,\Lambda}]$

For the analysis of the case $\lambda = \Lambda$ for the bidirectional ring, we define the probabilities $p_{k,+,i}$ and $p_{k,-,i}$ which are defined analogously to the definitions in (65) and (66) and where the wavelength is omitted in the subscript as we are only considering wavelength Λ. With reasoning similar as above we obtain

$$p_{k,+,i} = \begin{cases} \frac{i}{\eta-1}\frac{\binom{k-1}{i-1}}{\binom{\eta-2}{i-1}}, & \text{if } k \le \eta/2 \\ \frac{i}{\eta-1}\frac{\binom{k-1}{i-1}}{\binom{\eta-2}{i-1}}\left(1 - \frac{\binom{2k-\eta-1}{i-1}}{\binom{k-1}{i-1}}\right), & \text{if } k > \eta/2. \end{cases} \quad (81)$$

and

$$p_{k,-,i} = \begin{cases} \frac{i}{\eta-1} \frac{\binom{\eta-k-1}{i-1}}{\binom{\eta-2}{i-1}} \left(1 - \frac{\binom{\eta-2k}{i-1}}{\binom{\eta-k-1}{i-1}}\right), & \text{if } k \leq \eta/2 \\ \frac{i}{\eta-1} \frac{\binom{\eta-k-1}{i-1}}{\binom{\eta-2}{i-1}}, & \text{if } k > \eta/2. \end{cases} \tag{82}$$

From these probabilities, we can evaluate the conditional expectation of the hop distance as

$$\begin{aligned} E[H_{\Lambda,\Lambda}|F_{\Lambda,\Lambda} = i] &= \Lambda \sum_{k=1}^{\lfloor \eta/2 \rfloor} [k \cdot p_{k,+,i} + (\eta - k) \cdot p_{k,-,i}] \\ &\quad + \Lambda \sum_{k=\lfloor \eta/2 \rfloor+1}^{\eta-1} [k \cdot p_{k,+,i} + (\eta - k) \cdot p_{k,-,i}]. \end{aligned} \tag{83}$$

Inserting the $p_{k,+,i}$ and $p_{k,-,i}$ in (83) and unconditioning analogous to (67) we obtain

$$\begin{aligned} E[H_{\Lambda,\Lambda}|F_{\Lambda,\Lambda} = i] &= \frac{\Lambda i}{(\eta-1)\binom{\eta-2}{i-1}} \\ &\quad \cdot \left\{ \sum_{k=1}^{\eta-1} \left(k \binom{k-1}{i-1} + (\eta-k) \binom{\eta-k-1}{i-1} \right) \right. \\ &\quad - \sum_{k=1}^{\lfloor \eta/2 \rfloor} (\eta-k) \binom{\eta-2k}{i-1} \\ &\quad \left. - \sum_{k=\lfloor \eta/2 \rfloor+1}^{\eta-1} k \binom{2k-\eta-1}{i-1} \right\} \end{aligned} \tag{84}$$

and

$$\begin{aligned} E[H_{\Lambda,\Lambda}] &= \sum_{l=1}^{N-1} \mu_l \sum_{i=1}^{\eta-1} \frac{\binom{\eta-1}{i}\binom{N-\eta}{l-i}}{\binom{N-1}{l}} \frac{\Lambda}{\binom{\eta-1}{i}} \\ &\quad \cdot \left\{ 2\sum_{k=1}^{\eta-1} k \binom{k-1}{i-1} - \sum_{k=1}^{\lfloor \eta/2 \rfloor} (\eta-k) \binom{\eta-2k}{i-1} \right. \\ &\quad \left. - \sum_{k=\lfloor \eta/2 \rfloor+1}^{\eta-1} k \binom{2k-\eta-1}{i-1} \right\}. \end{aligned} \tag{85}$$

After some algebraic simplifications which are detailed in Appendix C we obtain

$$\begin{aligned} E[H_{\Lambda,\Lambda}] &= 2N - 2N\Lambda \sum_{l=1}^{N-1} \frac{\mu_l}{l+1} \\ &\quad + 2\Lambda \sum_{l=1}^{N-1} \frac{\mu_l}{\binom{N-1}{l}} \binom{N-\eta}{l+1} - \Lambda \sum_{l=1}^{N-1} \frac{\mu_l}{\binom{N-1}{l}} \\ &\quad \cdot \left\{ \sum_{k=1}^{\lfloor \eta/2 \rfloor} (\eta-k) \binom{N-2k}{l-1} \right. \\ &\quad \left. + \sum_{k=\lfloor \eta/2 \rfloor+1}^{\eta-1} k \binom{N+2k-2\eta-1}{l-1} \right\}. \end{aligned} \tag{86}$$

Note that for the case of unicast traffic, i.e., $\mu_1 = 1$ and $\mu_l = 0$ for $l = 2, \ldots, N-1$, this reduces for even η (which we recall is defined as $\eta = N/\Lambda$) to $E[H_{\Lambda,\Lambda}] = N^2/(4(N-1)\Lambda)$ and (80) reduces for even η to $E[H_{\Lambda,\lambda}] = N^2/(4(N-1)\Lambda)$. (For odd η we obtain analogously $E[H_{\Lambda,\Lambda}] = N^2/[4(N-1)\Lambda] - \Lambda/[4(N-1)]$ and $E[H_{\Lambda,\lambda}] = \lambda/(N-1) + N^2/[4(N-1)\Lambda] - \Lambda/[4(N-1)]$ for $\lambda \leq \lfloor \Lambda/2 \rfloor$.) With these expected hop distances we obtain for even η (as well as odd η in conjunction with even Λ) from (13) the well-known result that the expected hop distance is $E[H_\lambda | H_\lambda > 0] = N^2/[4(N-1)]$ on the wavelength on which the destination of the unicast packet is homed. We obtain the corresponding nominal capacities as $\hat{C}_T = \hat{C}_R = \hat{C}_M = 8\Lambda(N-1)/N$; noting that in our model of the bidirectional ring WDM network there are Λ wavelength channels in each direction for a total of 2Λ wavelength channels in the network, the nominal capacity approaches four times the total number of wavelength channels in the network for a large number of nodes N.

VI. Evaluation of Maximum Ring Segment Utilization Probability for Unidirectional Ring

In this section we evaluate the maximum ring segment utilization probability $u_{\max}$ for the unidirectional ring network, which is used for the calculation of the effective multicast capacity C_M as defined in (36) and in turn the effective transmission capacity C_T and reception capacity C_R as defined in (39).

Recall from Section III-F that $u_{\max}$ is defined as the largest of the probabilities $u(\lambda, s)$ that a given arbitrary multicast from a given arbitrary sending node traverses (utilizes) the ring segment s on wavelength λ, i.e., $u_{\max} = \max_{s,\lambda}\{u(\lambda, s)\}$. Also, recall that for the considered unidirectional ring network, the maximum utilization probability $u_{\max}$ is attained on every wavelength λ, $\lambda = 1, \ldots, \Lambda$, for every segment s that corresponds to a node on wavelength λ, i.e., for segments $s = \lambda + k \cdot \Lambda$, $k = 0, \ldots, \eta - 1$, on wavelength λ. Note that there are a total of $\Lambda \cdot \eta = N$ segments in the network that attain the maximum utilization probability $u_{\max}$.

To evaluate $u_{\max}$ we consider a given multicast transmission by node N and average the probabilities with which this multicast transmission utilizes the ring segments that attain the maximum utilization probability. More specifically, we have

$$u_{\max} = \frac{1}{N} \sum_{\lambda=1}^{\Lambda} \sum_{k=0}^{\eta-1} P(\text{segment } \lambda + k \cdot \Lambda \text{ on wavelength } \lambda \text{ is utilized} \mid \text{multicast by node } N) \tag{87}$$

$$= \frac{1}{N} \sum_{\lambda=1}^{\Lambda} \sum_{k=0}^{\eta-1} P(H_{\Lambda,\lambda} \geq \lambda + k \cdot \Lambda) \tag{88}$$

$$= \frac{1}{N} \sum_{\lambda=1}^{\Lambda} \sum_{k=0}^{\eta-1} \left(1 - \sum_{i=0}^{\eta} P(H_{\Lambda,\lambda} < \lambda + k \cdot \Lambda | F_{\Lambda,\lambda} = i) \cdot P(F_{\Lambda,\lambda} = i)\right). \tag{89}$$

For convenience we define

$$u_{\max} = \frac{1}{N}\sum_{\lambda=1}^{\Lambda} S_\lambda \tag{90}$$

and proceed to evaluate the summation terms S_λ for $\lambda = 1, \dots, \Lambda$.

For $\lambda = 1, \dots, \Lambda - 1$ we obtain analogous to (53) and (54)

$$S_\lambda = \sum_{k=0}^{\eta-1}\left(1 - \sum_{i=0}^{\eta}\frac{\binom{k}{i}}{\binom{\eta}{i}}\sum_{l=1}^{N-1}\mu_l\frac{\binom{\eta}{i}\binom{N-\eta-1}{l-i}}{\binom{N-1}{l}}\right) \tag{91}$$

$$= \eta - \sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}}\sum_{k=0}^{\eta-1}\binom{N+k-\eta-1}{l} \tag{92}$$

$$= \eta - \sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}}\left\{\binom{N-1}{l+1} - \binom{N-\eta-1}{l+1}\right\} \tag{93}$$

where (92) follows from the properties of the hypergeometric distribution. For $\lambda = \Lambda$ we obtain

$$S_\Lambda = \sum_{k=0}^{\eta-1} P(H_{\lambda,\Lambda} \geq (k+1)\cdot\Lambda) \tag{94}$$

$$= \sum_{k=0}^{\eta-1} P(H_{\Lambda,\Lambda} > k\cdot\Lambda) \tag{95}$$

$$= \frac{E[H_{\Lambda,\Lambda}]}{\Lambda} \tag{96}$$

$$= \eta - 1 - \sum_{l=1}^{N-1}\mu_l\frac{\binom{N-1}{l+1} - \binom{N-\eta}{l+1}}{\binom{N-1}{l}} \tag{97}$$

where (97) follows from (51). Inserting (93) and (97) in (90) we obtain

$$u_{\max} = 1 - \frac{1}{N} - \frac{1}{N}\sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}} \cdot\left\{\Lambda\binom{N-1}{l+1} - (\Lambda-1)\binom{N-\eta-1}{l+1} - \binom{N-\eta}{l+1}\right\}. \tag{98}$$

For the special case of unicast traffic, i.e., $\mu_1 = 1$ and $\mu_l = 0$ for $l = 2, \dots, N-1$, we obtain the effective capacities

$$C_M = C_T = C_R = 2\cdot\Lambda\cdot\frac{N-1}{N+\Lambda-2} \tag{99}$$

which are bounded from above by the corresponding nominal capacities. Note that the effective capacities approach the nominal capacities, i.e., twice the number of wavelength channels in the network, for a large number of nodes N.

VII. Evaluation of Maximum Ring Segment Utilization Probability for Bidirectional Ring

In this section we evaluate the maximum ring segment utilization probability $u_{\max} = \max_{s,\lambda}\{u(\lambda,+,s), u(\lambda,-,s)\}$ for the bidirectional ring network, which is the basis for evaluating the effective multicast capacity C_M(36) and the effective transmission and reception capacities C_T and C_R(39) of the bidirectional ring network. In the bidirectional ring network the packet copy on a wavelength λ is sent in the direction that gives the smaller hop distance; in the case of a tie, the packet copy is sent in either direction with equal probability. We note that the maximum utilization probability $u_{\max}$ is attained on each wavelength λ in the clockwise direction of each segment s that leads to a node on wavelength λ, i.e., for segments $s = \lambda + k\cdot\Lambda$, $k = 0, \dots, \eta - 1$, in the clockwise direction on wavelength λ. Analogously, the maximum utilization probability $u_{\max}$ is attained on each wavelength λ in the counter clockwise direction on each segment that leads to a node on λ.

Similar to the unidirectional ring, we evaluate $u_{\max}$ by considering a multicast by node N. Considering the N segments that attain $u_{\max}$ in the clockwise direction, we obtain equations (100) and (101) at the bottom of the page. We define for convenience

$$Nu_{\max} = \sum_{\lambda=1}^{\Lambda} S_\lambda \tag{102}$$

and proceed to evaluate the summation terms S_λ for $\lambda = 1, \dots, \Lambda$.

On the wavelength $\lambda = \Lambda$ there is a tie for the shortest path and the multicast packet is sent in the clockwise direction with probability one half. The tie occurs because both direct neighbor nodes of node N on wavelength Λ, namely nodes Λ and $N - \Lambda$, are equidistant from node N, i.e., have the same "offset" from node N. Thus,

$$S_\Lambda = \frac{1}{2}\sum_{k=0}^{\eta-1} P(H_{\Lambda,\Lambda} \geq (k+1)\cdot\Lambda) \tag{103}$$

$$= \frac{1}{2}\sum_{k=0}^{\eta-1} P(H_{\Lambda,\Lambda} > k\cdot\Lambda) \tag{104}$$

$$= \frac{1}{2\Lambda}E[H_{\Lambda,\Lambda}]. \tag{105}$$

For $\lambda = 1, \dots, \Lambda - 1$ we obtain with D_λ, $D_\lambda \in \{+, -, 0\}$, being a random variable that denotes the direction ($+\hat{=}$ clock-

$$u_{\max} = \frac{1}{N}\sum_{\lambda=1}^{\Lambda}\sum_{k=0}^{\eta-1} P(\text{segm. } \lambda + k\cdot\Lambda \text{ utilized on wavel. } \lambda \text{ in clockwise dir.} \mid \text{multicast by node } N) \tag{100}$$

$$= \frac{1}{N}\sum_{\lambda=1}^{\Lambda}\sum_{k=0}^{\eta-1} P(\text{multicast pkt. travels on wavel. } \lambda \text{ in clockwise dir. and } H_{\Lambda,\lambda} \geq \lambda + k\cdot\Lambda). \tag{101}$$

wise on wavelength λ, $-\hat{=}$ counter clockwise on λ, $0\hat{=}$ no copy transmission on λ) of the considered transmission by node N

$$S_\lambda = \sum_{k=0}^{\eta-1} P(H_{\Lambda,\lambda} \geq \lambda + k \cdot \Lambda \text{ and } D_\lambda = +) \quad (106)$$

$$= \sum_{k=0}^{\eta-1} P(H_{\Lambda,\lambda} + \Lambda - \lambda \geq (k+1) \cdot \Lambda \text{ and } D_\lambda = +) \quad (107)$$

$$= \sum_{k=0}^{\eta-1} P(H_{\Lambda,\lambda} + (\Lambda - \lambda) \cdot 1_{\{H_{\Lambda,\lambda}>0\}} \geq (k+1) \cdot \Lambda \text{ and } D_\lambda = +) \quad (108)$$

$$= \frac{1}{\Lambda} E\left[\left(H_{\Lambda,\lambda} + (\Lambda-\lambda)\cdot 1_{\{H_{\Lambda,\lambda}>0\}}\right)\cdot 1_{\{D_\lambda=+\}}\right] \quad (109)$$

$$= \frac{1}{\Lambda} \cdot E\left[H_{\Lambda,\lambda} \cdot 1_{\{D_\lambda=+\}}\right] + \frac{\Lambda-\lambda}{\Lambda} \cdot P(D_\lambda = +). \quad (110)$$

We observe that

$$E\left[H_{\Lambda,\lambda} \cdot 1_{\{D_\lambda=+\}}\right] + E\left[H_{\Lambda,\lambda} \cdot 1_{\{D_\lambda=-\}}\right] = E[H_{\Lambda,\lambda}] \quad (111)$$

and that by symmetry

$$E\left[H_{\Lambda,\lambda} \cdot 1_{\{D_\lambda=+\}}\right] = E\left[H_{\Lambda,\Lambda-\lambda} \cdot 1_{\{D_{\Lambda-\lambda}=-\}}\right]. \quad (112)$$

For even Λ, which are commonly used in practice and which we consider in the following unless otherwise noted, we find it convenient to break the group of wavelengths $\lambda = 1, \ldots, \Lambda - 1$ into the three groups (a) $\lambda = 1, \ldots, \lfloor \frac{\Lambda-1}{2} \rfloor$, (b) $\lambda = \Lambda/2$, and (c) $\lambda = \lfloor \frac{\Lambda+1}{2} \rfloor, \ldots, \Lambda - 1$. (Only groups (a) and (c) are considered for odd Λ.)

We obtain for $\lambda = \Lambda/2$

$$S_{\Lambda/2} = \frac{1}{\Lambda} \cdot E\left[H_{\Lambda,\Lambda/2} \cdot 1_{\{D_{\Lambda/2}=+\}}\right] + \frac{1}{2} \cdot P(D_{\Lambda/2} = +) \quad (113)$$

$$= \frac{1}{2\Lambda} E\left[H_{\Lambda,\Lambda/2}\right] + \frac{1}{4} P(F_{\Lambda,\Lambda/2} > 0) \quad (114)$$

whereby (114) follows from the symmetry properties.

We have now all the summation terms required to evaluate $Nu_{\max}$ (102). To simplify the ensuing analysis we "map" the wavelengths $\lambda = \lfloor \frac{\Lambda+1}{2} \rfloor, \ldots, \Lambda - 1$ to the wavelengths $\lambda = 1, \ldots, \lfloor \frac{\Lambda-1}{2} \rfloor$ using the symmetry property (112) and obtain

$$\begin{aligned} Nu_{\max} &= \sum_{\lambda=1}^{\lfloor(\Lambda-1)/2\rfloor} \left\{ \frac{1}{\Lambda} E[H_{\Lambda,\lambda} \cdot 1_{\{D_\lambda=+\}}] + \frac{\Lambda-\lambda}{\Lambda} \cdot P(D_\lambda = +) \right\} \\ &+ \frac{1}{2\Lambda} E\left[H_{\Lambda,\Lambda/2}\right] + \frac{1}{4} P(F_{\Lambda,\Lambda/2} > 0) \\ &+ \sum_{\lambda=1}^{\lfloor(\Lambda-1)/2\rfloor} \left\{ \frac{1}{\Lambda} E[H_{\Lambda,\lambda} \cdot 1_{\{D_\lambda=-\}}] + \frac{\lambda}{\Lambda} \cdot P(D_\lambda = -) \right\} \\ &+ \frac{1}{2\Lambda} E[H_{\Lambda,\Lambda}] \end{aligned} \quad (115)$$

$$\begin{aligned} &= \frac{1}{2\Lambda} E[H_{\Lambda,\Lambda}] \\ &+ \sum_{\lambda=1}^{\lfloor(\Lambda-1)/2\rfloor} \left\{ \frac{1}{\Lambda} E[H_{\Lambda,\lambda}] + \left(1 - \frac{\lambda}{\Lambda}\right) \cdot \delta^+ + \frac{\lambda}{\Lambda} \cdot \delta^- \right\} \\ &+ \frac{1}{2\Lambda} E\left[H_{\Lambda,\Lambda/2}\right] + \frac{1}{4} P(F_{\Lambda,\Lambda/2} > 0) \end{aligned} \quad (116)$$

$$= \frac{1}{2\Lambda} \sum_{\lambda=1}^{\Lambda} E[H_{\Lambda,\lambda}] + \sum_{\lambda=1}^{\lfloor(\Lambda-1)/2\rfloor} \left\{ \left(1 - \frac{\lambda}{\Lambda}\right) \cdot \delta^+ + \frac{\lambda}{\Lambda} \cdot \delta^- \right\} + \frac{1}{4} P(F_{\Lambda,\Lambda/2} > 0). \quad (117)$$

We remark that (116) is obtained by exploiting the property (111). We also remark that $\delta^+ = P(D_\lambda = +)$ denotes the probability that the multicast packet is transmitted in the clockwise direction on a wavelength $\lambda < \Lambda/2$ (which is independent of λ). Analogously, $\delta^- = P(D_\lambda = -)$ denotes the probability that the packet is transmitted in the counter clockwise direction.

To further simplify this expression we exploit the following two properties. First, that δ^+ and δ^- are independent of λ as long as $\lambda < \Lambda/2$. Second, we note that if a copy of the considered multicast packet is transmitted on wavelength λ (which is the case with probability $1 - P(F_{\Lambda,\lambda} = 0)$), then it is transmitted either in the clockwise $(+)$ or counterclockwise $(-)$ direction. Hence

$$\delta^+ + \delta^- = 1 - P(F_{\Lambda,\lambda} = 0) \quad (118)$$

i.e., $\delta^- = 1 - P(F_{\Lambda,\lambda} = 0) - \delta^+$. Furthermore, we obtain by retracing the analysis leading to (69) with $i = 0$

$$1 - P(F_{\Lambda,\lambda} = 0) = 1 - \sum_{l=1}^{N-1} \mu_l \cdot \frac{\binom{N-\eta-1}{l}}{\binom{N-1}{l}} =: f_0. \quad (119)$$

With these properties we can simplify the expression for $Nu_{\max}$ as follows

$$\begin{aligned} Nu_{\max} &= \frac{1}{2\Lambda} \sum_{\lambda=1}^{\Lambda} E[H_{\Lambda,\lambda}] + \delta^+ \sum_{\lambda=1}^{\lfloor(\Lambda-1)/2\rfloor} \left(1 - \frac{2\lambda}{\Lambda}\right) \\ &+ \left(\frac{1}{4} + \sum_{\lambda=1}^{\lfloor(\Lambda-1)/2\rfloor} \frac{\lambda}{\Lambda}\right) f_0 \end{aligned} \quad (120)$$

$$= \frac{1}{2\Lambda} \sum_{\lambda=1}^{\Lambda} E[H_{\Lambda,\lambda}] + \delta^+ \left\lfloor \frac{\Lambda-1}{2} \right\rfloor \left(1 - \frac{1}{\Lambda} \left\lfloor \frac{\Lambda+1}{2} \right\rfloor\right) + f_0 \left(\frac{\lfloor \frac{\Lambda-1}{2} \rfloor \cdot \lfloor \frac{\Lambda+1}{2} \rfloor}{2 \cdot \Lambda} + \frac{1}{4} \right) \quad (121)$$

$$= \frac{1}{2\Lambda} \sum_{\lambda=1}^{\Lambda} E[H_{\Lambda,\lambda}] + \frac{\delta^+}{2} \left(\frac{\Lambda}{2} - 1\right) + \frac{\Lambda}{8} f_0 \quad (122)$$

whereby the mean hop distance $E[H_{\Lambda,\lambda}]$ for $\lambda = 1, \ldots, \Lambda/2$ is given by (80), the mean hop distance $E[H_{\Lambda,\lambda}]$ for $\lambda = \Lambda/2 + 1, \ldots, \Lambda - 1$ is given by the symmetry property (64) in conjunction with (80), and the mean hop distance $E[H_{\Lambda,\Lambda}]$ is given by (86). It remains to evaluate the probability δ^+.

A. Evaluation of Probability δ^+ of Sending in Clockwise Direction

Throughout the derivation of δ^+ we consider only $\lambda < \Lambda/2$ as required for the evaluation of $u_{\max}$. Recall from Section V that $p_{\lambda,\lambda+k\cdot\Lambda,+,i}$ defined in (65) is the probability that given there are i multicast destinations on λ, the highest indexed multicast destination is node $\lambda + k \cdot \Lambda$ and the multicast copy is transmitted in clockwise direction. Thus

$$\delta^+ = \sum_{i=1}^{\eta}\sum_{k=0}^{\eta} p_{\lambda,\lambda+k\cdot\Lambda,+,i} \cdot P(F_{\Lambda,\lambda} = i). \tag{123}$$

Combining the two expressions (73) and (77) derived in Section V for $p_{\lambda,\lambda+k\cdot\Lambda,+,i}$ for the two cases where 1) the highest indexed node lies at most halfway around the ring and 2) the highest indexed node lies more than halfway around the ring $(k > \eta/2)$, we obtain

$$\begin{aligned}\delta^+ &= \sum_{i=1}^{\eta}\left(\sum_{k=0}^{\eta-1}\frac{i}{\eta}\cdot\frac{\binom{k}{i-1}}{\binom{\eta-1}{i-1}} - \sum_{k=\lfloor(\eta+1)/2\rfloor}^{\eta-1}\frac{i}{\eta}\cdot\frac{\binom{2k-\eta}{i-1}}{\binom{\eta-1}{i-1}}\right) \\ &\quad \cdot P(F_{\Lambda,\lambda} = i) \qquad (124)\\ &= \sum_{i=1}^{\eta}\left(\sum_{k=0}^{\eta-1}\frac{\binom{k}{i-1}}{\binom{\eta}{i}} - \sum_{k=\lfloor(\eta+1)/2\rfloor}^{\eta-1}\frac{\binom{2k-\eta}{i-1}}{\binom{\eta}{i}}\right) \\ &\quad \cdot P(F_{\Lambda,\lambda} = i) \qquad (125)\\ &= \sum_{i=1}^{\eta}\left(\frac{\sum_{j=0}^{\lfloor(\eta-1)/2\rfloor}\binom{\eta-1-2j}{i-1}}{\binom{\eta}{i}}\right)\cdot P(F_{\Lambda,\lambda} = i) \qquad (126)\\ &= \sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}}\sum_{j=0}^{\lfloor(\eta-1)/2\rfloor}\sum_{i=1}^{\eta}\binom{\eta-1-2j}{i-1} \\ &\quad \times \binom{N-1-\eta}{l-i} \qquad (127)\\ &= \sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}}\sum_{j=0}^{\lfloor(\eta-1)/2\rfloor}\binom{N-2j-2}{l-1} \qquad (128)\end{aligned}$$

where $P(F_{\Lambda,\lambda} = i)$ follows analogous from (54) and (128) follows from the properties of the hypergeometric distribution.

For the special case of broadcast traffic with $\mu_l = 0$ for $l = 1, \ldots, N-2$ and $\mu_{N-1} = 1$, (128) gives $\delta^+ = 1$. This is intuitive since the hop distances required to reach all nodes on the considered wavelengths $\lambda < \Lambda/2$ in the clockwise direction are smaller (due to the smaller offset of λ hops of the first node on the wavelength from node N) than in the counter clockwise direction.

For the special case of unicast traffic with $\mu_1 = 1$ and $\mu_l = 0$ for $l = 2, \ldots, N-1$, (128) gives $\delta^+ = [\lfloor(\eta-1)/2\rfloor + 1]/(N-1)$. Recalling from the end of Section V the mean hop distances for unicast traffic, we obtain from (122) in conjunction with (36) the effective multicast capacity for unicast traffic for even Λ as

$$C_M = 8\Lambda\frac{N(N-1)}{N(N+\Lambda) + 2\Lambda(\Lambda-2)\left\lfloor\frac{N+\Lambda}{2\Lambda}\right\rfloor} \tag{129}$$

which for a large number of nodes N approaches four times the number of wavelength channels in the bidirectional ring network (recall that there are Λ wavelength channels in each direction in the bidirectional ring network).

VIII. Numerical Results

In this section, we compare the nominal and effective capacities and examine the impact of the number of nodes N and the multicast fanout F on the effective transmission, reception, and multicast capacities of both unidirectional and bidirectional ring WDM networks. We consider different unicast, multicast, broadcast, and mixed traffic scenarios. In all subsequent numerical investigations we set the number of wavelength channels to $\Lambda = 8$ in the single-fiber WDM ring network, and to $\Lambda = 4$ wavelength channels on each of the two counter-directional fibers of the dual-fiber WDM ring network. Thus the total number of wavelength channels is the same, namely eight, in each network.

A. Comparison of Nominal and Effective Capacities

In Table II we compare the nominal transmission, reception, and multicast capacities with the effective transmission, reception, and multicast capacities. We consider the extreme traffic scenarios of unicast traffic for which $\mu_1 = 1$ and $\mu_l = 0$ for $l = 2, \ldots, N-1$ and broadcast traffic for which $\mu_l = 0$ for $l = 1, \ldots, N-2$ and $\mu_{N-1} = 1$. We observe from the table that in general the nominal capacity provides an upper bound for the corresponding effective capacity and for a large number of nodes N the bound becomes rather tight. The intuitive explanation for this behavior is as follows. There are two main effects leading to a difference between the nominal and effective capacity. The first effect appears only for $N = \Lambda$. For this scenario there is only a single node $(\eta = 1)$ on a given wavelength. As a consequence there is only limited spatial wavelength reuse possible since the node homed on a given wavelength can not immediately reuse the wavelength freed up by destination stripping, simply because there are no eligible destinations downstream on the wavelength. The second effect is due to relative alignment of the paths taken by the packet copies on a given wavelength and the resulting uneven loading of the ring segments. Intuitively, only every Λth node strips (removes) traffic from a given wavelength channel but all nodes—including the node that just stripped the traffic destined to it and the $\Lambda - 1$ nodes encountered in the direction of the fiber before the next node homed on the channel—send traffic onto the wavelength channel. As a result of the traffic sent by the $\Lambda - 1$ nodes inbetween the two nodes homed on the channel, the ring segments become more heavily loaded as we approach a node homed on the channel. In particular, the segment directly connecting to the homed node experiences the heaviest loading and hence the maximum link utilization probability $u_{\max}$. As the number of network nodes N increases for a fixed number of wavelength channels Λ, the relative contribution of the $\Lambda - 1$ nodes inbetween two nodes homed on the channel toward the total traffic load diminishes. Effectively, for larger N the loading of the ring segments is smoothed out, the effect of the relative alignment of the paths travelled by the packet copies on the wavelength channel diminishes, and the effective capacity approaches the

TABLE II
NOMINAL AND EFFECTIVE CAPACITIES FOR UNICAST AND BROADCAST TRAFFIC FOR DIFFERENT NUMBERS OF NODES N

Unidirectional Ring, $\Lambda = 8$ wavelengths

	Unicast Traffic		Broadcast Traffic					
	Nom. Cap.	Eff. Cap.	Nominal Capacity			Effective Capacity		
N	$\hat{C}_T = \hat{C}_R = \hat{C}_M$	$C_T = C_R = C_M$	$\hat{C}_T$	$\hat{C}_R$	$\hat{C}_M$	C_T	C_R	C_M
8	16	8.00	16.00	16.00	2.286	8.00	8	1.143
16	16	10.91	11.13	20.87	1.391	8.53	16	1.067
32	16	13.05	9.31	36.07	1.164	8.26	32	1.032
64	16	14.40	8.61	67.76	1.076	8.13	64	1.016
240	16	15.54	8.15	243.57	1.019	8.03	240	1.004

Bidirectional Ring, $\Lambda = 4$ wavelengths in each direction

	Unicast Traffic		Broadcast Traffic					
	Nom. Cap.	Eff. Cap.	Nominal Capacity			Effective Capacity		
N	$\hat{C}_T = \hat{C}_R = \hat{C}_M$	$C_T = C_R = C_M$	$\hat{C}_T$	$\hat{C}_R$	$\hat{C}_M$	C_T	C_R	C_M
4	24.00	8.00	24.00	24.00	8.00	8.00	8	2.667
8	28.00	16.00	12.80	22.40	3.200	9.14	16	2.286
16	30.00	21.82	9.85	36.92	2.462	8.53	32	2.133
32	31.00	26.11	8.83	68.41	2.207	8.26	64	2.065
64	31.50	28.80	8.39	132.20	2.098	8.13	128	2.032
240	31.87	31.09	8.10	484.05	2.025	8.03	480	2.008

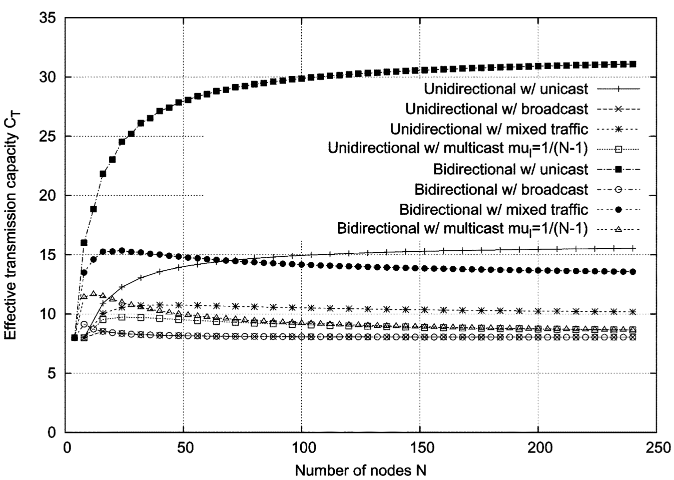

Fig. 2. Effective transmission capacity C_T of uni- and bidirectional WDM rings versus number of nodes N for unicast, multicast, mixed, and broadcast traffic with $\Lambda = 8$ wavelengths on fiber in unidirectional ring and $\Lambda = 4$ wavelengths on each fiber in bidirectional ring, for a total of eight wavelengths in each network.

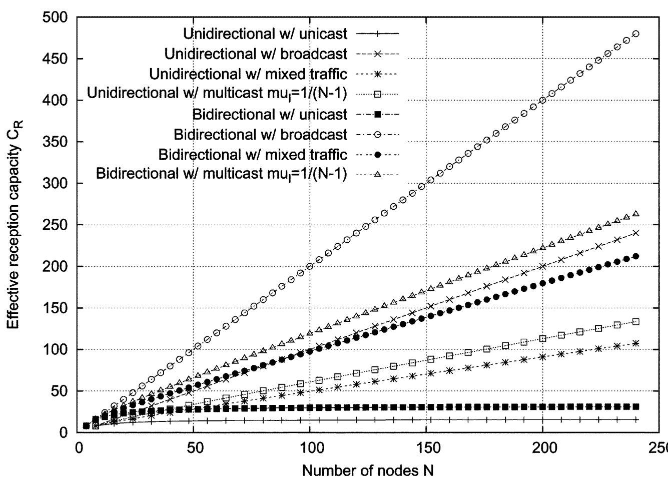

Fig. 3. Effective reception capacity C_R of uni- and bidirectional WDM rings versus number of nodes N for networks with a total of eight wavelengths.

nominal capacity. Our numerical results in Table II indicate that for the considered typical scenario with eight wavelength channels in the network, for $N = 64$ or more nodes the nominal capacity overestimates the effective capacity by less than 10%.

We also note that the nominal capacity does not take the number of receivers in a node into consideration. In the considered unidirectional ring network each node has one receiver, hence the actually achievable receiver throughput C_R can not exceed the number of nodes N; in the bidirectional ring network every node has two receivers limiting the receiver throughput to no more than $2N$. We observe from Table II that for the broadcast traffic scenario the nominal reception capacity $\hat{C}_R$ exceeds the limitations due to the number of available receivers (and correspondingly of the heaviest loaded links leading to the receivers). The effective capacity, on the other hand, considers the loading on the heaviest loaded ring segments leading to the individual receivers and permits a maximum loading level (utilization probability) of one. As a consequence the effective reception capacities C_R for broadcast traffic are exactly equal to the numbers of available receivers, as observed from the table.

B. Impact of Number of Nodes on Effective Capacities

Figs. 2–4 depict the effective transmission capacity C_T, effective reception capacity C_R, and effective multicast capacity C_M of unidirectional and bidirectional WDM rings versus number of nodes N for unicast traffic, broadcast traffic, multicast traffic (with $\mu_l = 1/(N-1)$ for $l = 1, \ldots, N-1$), and a mixed traffic scenario (with $\mu_1 = 0.8$ and $\mu_l = 0.2/(N-2)$ for $l = 2, \ldots, N-1$). In our discussion of these plots we focus initially on the unicast and broadcast traffic and turn then to the multicast and mixed traffic.

For unicast traffic, we observe from Fig. 2 that the effective transmission capacities approach the limit of twice the number of wavelength channels in the unidirectional ring and four times the number of wavelength channels in the bidirectional ring.

For the broadcast and multicast traffic we observe from Fig. 2 that the effective transmission capacities approach eight for a large number of nodes N. This is due to the fact that with in-

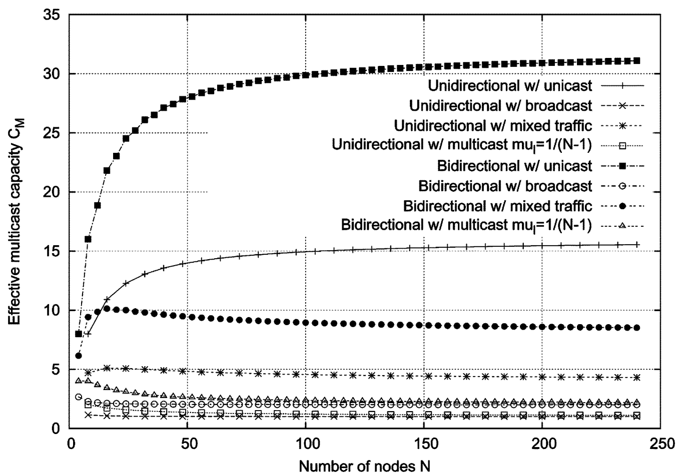

Fig. 4. Effective multicast capacity C_M of uni- and bidirectional WDM rings versus number of nodes N for networks with a total of eight wavelengths.

creasing N (and fixed Λ) more nodes η share a given drop wavelength and more nodes are involved in forwarding (multihopping) packets which prevents them from transmitting their own locally generated packets. (Recall from Section III that packets forwarded by nodes do not contribute to C_T.) As a consequence of this increasing forwarding burden the transmission capacity C_T decreases with increasing N. As shown in Fig. 2, for large N the effective transmission capacities asymptotically approach the number of wavelength channels in both uni- and bidirectional WDM rings since each packet copy visits all nodes on each wavelength. As a result, with increasing N, wavelengths can be spatially reused on increasingly smaller ring segments and the capacity improvement due to spatial wavelength reuse diminishes.

We observe from Fig. 3 that for unicast traffic the effective reception capacity C_R of both ring networks is identical to the respective transmission capacity C_T. Clearly, this is because for unicast traffic each packet is received by a single destination node. In contrast, for broadcast traffic, C_R is significantly larger than C_T. Note that C_R in the bidirectional ring is essentially twice as large as that in the unidirectional ring. This is primarily due to the duplicated node structure for each fiber in the bidirectional ring, where each node has two FR's, one each tuned to the node's drop wavelength on each of the two fibers. This allows the broadcast reception capacity to be exactly equal to twice the number of nodes in the bidirectional ring network. In both networks, C_R grows linearly with increasing N. This is due to the fact that in both networks each broadcast packet is received by $(N-1)$ nodes. Similarly, for multicast traffic C_R grows approximately linearly with increasing N but is significantly smaller than the reception capacity obtained for broadcast traffic. Clearly, compared to broadcast traffic fewer destination nodes receive a packet for multicast traffic, leading to a smaller reception capacity. We also observe that for multicast traffic the bidirectional ring outperforms the unidirectional ring in terms of C_R. This is primarily due to the fact that a wavelength in the bidirectional ring homes twice as many nodes as a wavelength in the unidirectional ring resulting in approximately twice the C_R in the bidirectional ring for the same level of C_T. Consequently, more destination receivers are active resulting in an increased reception capacity C_R.

We observe from Fig. 4 that for broadcast and multicast traffic the effective multicast capacity C_M of both ring networks is significantly smaller than for unicast traffic and decreases with increasing N. This is because generally each multicast/broadcast packet requires multiple transmissions of packet copies on different wavelengths and with increasing N the transmission capacity C_T decreases (see Fig. 2). For multi- and broadcast traffic, the C_M of the bidirectional ring approaches two for large N, and one in the unidirectional ring. This is because packets destined to a large number of nodes essentially require copy transmissions around the entire ring on all wavelengths. With two sets of wavelengths in the bidirectional ring (whereby each wavelength is homing twice the number of nodes than a wavelength in the unidirectional ring), the bidirectional ring allows in the limit for two simultaneous multicasts.

Turning our attention to the mixed traffic scenario and focusing for now on the bidirectional ring network, we observe that the effective transmission and multicast capacities initially increase, reach a maximum, and then decrease slightly as the number of nodes increases. There are mainly two effects at work. When starting from a small number of nodes N, the increasing N initially smoothes out the loading of the ring segments as discussed above and thus permits for a higher multicast throughput and correspondingly higher transmitter and receiver throughput. As N increases further, the effect of having to transmit the multicast packet copies essentially all around the ring to reach all intended destinations leads to a slight decrease in the effective multicast capacity and corresponding transmission capacity while the reception capacity continues to increase due to the larger number of destination nodes. More specifically, for a large number of nodes the effective multicast capacity converges toward a value close to eight. The intuitive explanation for this behavior is as follows. In the considered traffic mix scenario, 80% of the traffic is unicast traffic which requires a mean hop distance of approximately $N/4$ in the bidirectional ring network. 20% of the traffic is multicast traffic which for a large number of nodes requires copy transmissions around the entire ring on all wavelengths, i.e., a mean hop distance of close to $N\Lambda$. Thus, the overall mean hop distance required to serve a packet of the considered traffic mix tends to $N\Lambda[0.2 + 0.8/(4\Lambda)]$. Using the last expression in (28) we obtain the corresponding multicast capacity as $2/[0.2 + 0.2/\Lambda] = 8$. By an analogous argument the effective multicast capacity C_M for mixed traffic in the unidirectional ring tends to $N\Lambda/[0.2N\Lambda + 0.8N/2] = 4$ for a large number of nodes N.

Another interesting observation from Figs. 2–4 is that for a few scenarios the multicast capacity is continuously decreasing for increasing N, while the corresponding transmission capacity initially increases, peaks, and then decreases for increasing N, as for instance clearly visible for the multicast traffic in the bidirectional ring network. The explanation for this behavior is as follows. Initially, the increasing N and correspondingly increasing number of destination nodes require the transmission of an increasing number of multicast packet copies, i.e.,

$E[\Delta]$ increases. This effect of an increased number of packet copy transmission and increased transmission capacity dominates over the underlying trend of decreasing multicast capacity. As N is further increased, the packet copies tend to travel longer hop distances on the individual wavelength channels, until each packet copy travels essentially all around the ring, resulting in decreasing multicast and transmission capacities.

C. Impact of Multicast Fanout on Effective Capacities

Next, we investigate the impact of the maximum fanout $F_{\max}$ of multicast packets on the effective transmission, reception, and multicast capacity of uni- and bidirectional WDM rings. In the following, we consider $N = 128$ nodes and vary the maximum fanout of multicast packets in the interval $2 \leq F_{\max} \leq 127$. We examine different mixes of unicast and multicast traffic. Specifically, the fraction of unicast traffic is given by $\mu_1 \in \{0, 0.5, 1.0\}$ and the fraction of multicast traffic with fanout l is given by $\mu_l = (1 - \mu_1)/(F_{\max} - 1)$ for $2 \leq l \leq F_{\max}$; and $\mu_l = 0$ otherwise, i.e., for $F_{\max} < l \leq 127$.

Fig. 5 depicts the effective transmission capacity C_T of unidirectional and bidirectional ring WDM networks versus maximum fanout $F_{\max}$ for different fraction of unicast traffic $\mu_1 \in \{0, 0.5, 1.0\}$. For unicast-only traffic ($\mu_1 = 1.0$) the effective transmission capacity of unidirectional ring and bidirectional ring equals 15.16 and 30.33, respectively, independent from $F_{\max}$ since there is no multicast traffic. With $\mu_1 = 0.5$, i.e., 50% of the generated packets are unicast and the other 50% are multicast, the effective transmission capacity C_T of both WDM rings decreases with increasing maximum fanout $F_{\max}$. This is due to the forwarding burden which prevents nodes from transmitting locally generated traffic, leading to a decreased transmission capacity C_T. The forwarding burden on each node increases with increasing fanout $F_{\max}$ of multicast packets. With $\mu_1 = 0$ there is no unicast traffic and every generated multicast packet has a fanout of up to $F_{\max}$ destinations. We observe from Fig. 5 that for multicast-only traffic the transmission capacity C_T of both uni- and bidirectional ring networks is further decreased due to the increased forwarding burden. Note that the transmission capacity loss for multicast traffic is more pronounced in the bidirectional ring than in the unidirectional ring. This is mainly because the total mean hop distance in the unidirectional ring network increases from $N/2$ for unicast traffic to close to $N\Lambda$ for multicast traffic with a large fanout. On the other hand, in the bidirectional ring the total mean hop distance increases from approximately $N/4$ for unicast traffic to close to $N\Lambda$ for multicast traffic. As a result the drop (loss) in the transmission capacity is close to twice as large in the bidirectional ring compared to the unidirectional ring.

Fig. 6 depicts the effective reception capacity C_R of unidirectional and bidirectional ring WDM networks versus fanout $F_{\max}$ for different fraction of unicast traffic $\mu_1 \in \{0, 0.5, 1.0\}$. For unicast-only traffic we have $C_R = C_T$ in both ring networks since each packet is destined to a single receiver. With $\mu = 0.5$ half of the packets are multicast. With increasing maximum fanout $F_{\max}$ each multicast packet is destined to more receivers, resulting in an increased C_R compared to C_T. This effect is slightly further enhanced for multicast-only traffic ($\mu_1 = 0$) where each packet is destined to up to $F_{\max}$ receivers.

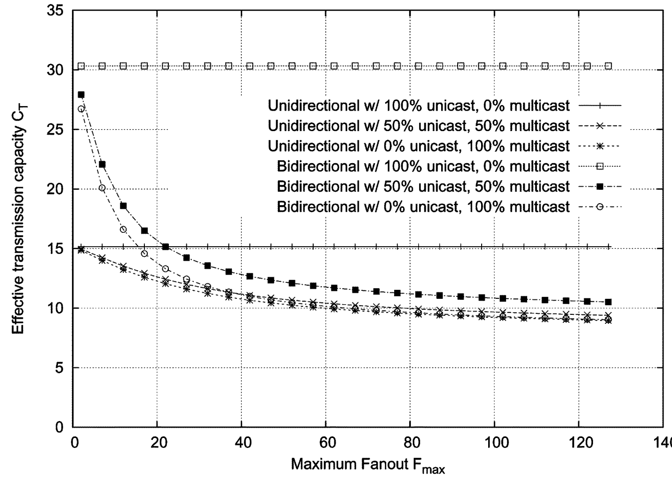

Fig. 5. Effective transmission capacity C_T of uni- and bidirectional WDM rings versus maximum fanout $F_{\max}$ for different fraction of unicast traffic $\mu_1 \in \{0, 0.5, 1.0\}$ with $N = 128$ nodes and a total of eight wavelengths.

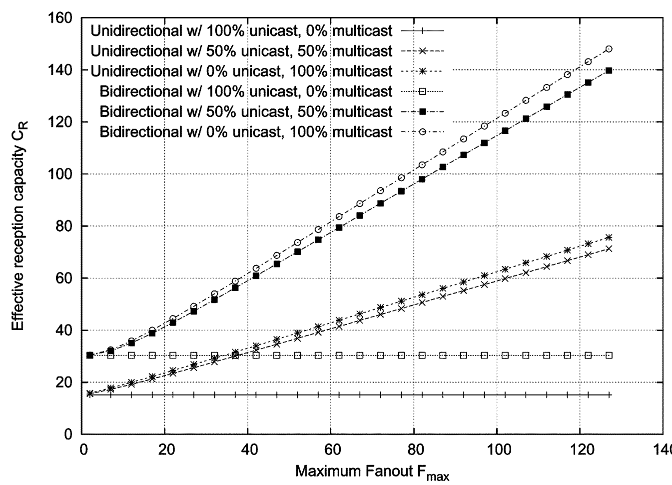

Fig. 6. Effective reception capacity C_R of uni- and bidirectional WDM rings versus fanout $F_{\max}$ for different fraction of unicast traffic $\mu_1 \in \{0, 0.5, 1.0\}$ with $N = 128$ nodes and a total of 8 wavelengths.

Fig. 7 depicts the multicast capacity C_M of unidirectional and bidirectional ring WDM networks versus maximum fanout $F_{\max}$ for different fraction of unicast traffic $\mu_1 \in \{0, 0.5, 1.0\}$. Clearly, for unicast traffic we have $C_M = C_T$ in both ring WDM networks. For $\mu_1 = 0.5$ and in particular $\mu_1 = 0$ the multicast capacity C_M of both ring networks is significantly smaller than that for unicast traffic and decreases with increasing $F_{\max}$. This is due to the fact that on average with increasing $F_{\max}$ more multicast copies must be sent on different wavelengths and C_T decreases with increasing $F_{\max}$ (see Fig. 5). Similar to C_T in Fig. 5, the multicast capacity loss for multicast traffic is more pronounced in the bidirectional ring than in the unidirectional ring.

Focusing on the unidirectional ring network, we observe from Fig. 5 that the effective transmission capacity with 100% multicast traffic is only slightly smaller than with 50% multicast traffic, whereas we observe from Fig. 6 that the effective receiver capacity for 100% multicast traffic is only slightly larger. The explanation for these slight changes in the transmitter and reception capacity despite the profound differences in the traffic

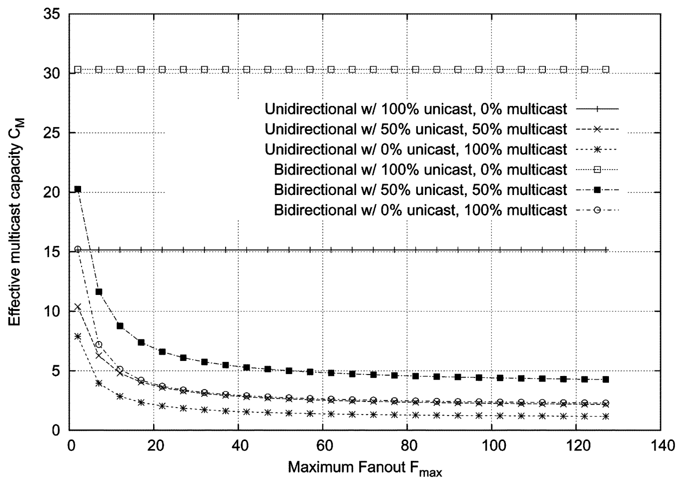

Fig. 7. Effective multicast capacity C_M of uni- and bidirectional WDM rings versus fanout $F_{\max}$ for different fraction of unicast traffic $\mu_1 \in \{0, 0.5, 1.0\}$ with $N = 128$ nodes and a total of 8 wavelengths.

pattern is as follows. As Fig. 7 indicates, the effective multicast capacity tends to a value close to one for 100% multicast traffic as the maximum multicast fanout increases; this is because the large fanouts require essentially packet copy transmissions around the entire ring on all wavelengths (i.e., a total hop count close to $N\Lambda$ to serve one multicast packet). For 50% multicast traffic, the effective multicast capacity tends to a value close to two for increasing multicast fanout; intuitively this is because one half of the traffic (namely the multicast traffic) requires a mean hop distance close to $N\Lambda$ and the other half (namely the unicast traffic) requires a mean hop distance of $N/2$, resulting in an overall mean hop distance of approximately $N\Lambda[1 + 1/(2\Lambda)]/2$ and hence a nominal multicast capacity of approximately $2/[1+1/(2\Lambda)]$. Thus, with 50% multicast traffic the maximum number of simultaneously ongoing packet transmissions (maximum multicast throughout) is roughly twice as large as with 100% multicast traffic. However, with 50% multicast traffic, half the traffic is due to multicast packets and the other half due to unicast packets. Thus, the multicast throughput due to multicast packets with 50% multicast traffic is roughly as large as—or more precisely only slightly less than—with 100% multicast traffic, i.e., the number of simultaneously ongoing multicasts at the capacity level is about the same with both traffic scenarios (more precisely only slightly less with 50% multicast traffic). Thus, the slightly larger transmission capacity with 50% multicast traffic is due to the contributions of the unicast packets (which require less multihopping than multicast packets) to the transmission capacity. The slightly larger reception capacity with 100% multicast traffic is due to the larger fanout of the slightly more ongoing multicast packet transmissions. The explanation for the bidirectional ring network is analogous, keeping in mind that a unicast packet requires a mean hop distance of approximately $N/4$ on the bidirectional ring.

IX. Conclusion

We have developed an analytical methodology for calculating the transmission, reception, and multicast capacities of unidirectional and bidirectional packet-switched ring WDM networks. These capacities give the maximum long run average transmitter, receiver, and multicast throughputs in the networks. We have evaluated both the nominal capacities which are based on the mean hop distances of the packet copies travelled on the individual wavelength channels and the effective capacities which are based on the utilization probability of the most heavily loaded ring segments and give the actually achievable capacities in the networks. We have provided explicit expressions for the capacities for a general multicast traffic model, which accommodates unicast, multicast, and broadcast traffic, and mixes thereof. We have also explicitly considered the capacities for unicast traffic, which are obtained as a special case of our multicast capacity analysis. In particular, we have shown that for unicast traffic in the unidirectional ring the mean hop distance is exactly half the node count and hence the nominal capacity is twice the number of wavelength channels. For unicast traffic in the bidirectional ring the mean hop distance is generally slightly larger than one fourth of the node count and approaches one fourth of the node count for a large number of nodes; correspondingly, the nominal capacity approaches from below four times the number of wavelength channels in the network. The effective capacities in both uni- and bidirectional ring WDM networks are bounded from above by the nominal capacities and approach the nominal capacities for a large number of nodes. The derived capacities can be used to evaluate and compare future multicast-capable MAC protocols for ring WDM networks in terms of transmitter, receiver, and multicast throughput efficiency.

For different unicast, multicast, and broadcast traffic scenarios we have numerically examined the impact of the number of ring nodes and the fanout of multicast packets on the capacity performance of both ring networks. The main findings of our numerical investigations are as follows. For broadcast traffic and for large multicast fanouts the packet forwarding burden on the ring nodes is high, resulting in a decreased transmission capacity. For an increasing number of nodes spatial wavelength reuse diminishes and the transmission capacity of both unidirectional and bidirectional ring networks asymptotically drops down to the number of wavelength channels. That is, for broadcast traffic and multicast traffic with large fanout the capacity improvement due to spatial wavelength reuse vanishes for an increasing number of ring nodes. We have also observed that for broadcast traffic, the reception capacity increases with increasing number of network nodes and multicast fanout. For multi- and broadcast traffic the bidirectional ring clearly outperforms the unidirectional ring in terms of reception capacity.

Interestingly, we have seen that the transmission capacity and multicast capacity loss for increasing fanout of multicast packets is more pronounced in the bidirectional ring than in the unidirectional ring. This is due to the fact that in the bidirectional ring network the total mean hop distance increases from approximately $N/4$ to $N\Lambda$ for increasing multicast fanout whereas it increases from $N/2$ to approximately $N\Lambda$ in the unidirectional ring. Note that we have assumed that in the bidirectional ring all multicast copies of a given multicast packet are sent in that direction which provides the smaller hop distance between source node and final multicast destination node. By transmitting multicast copies belonging to the same multicast packet in both directions rather than only in one direction this capacity loss could be somewhat mitigated. More precisely, both directions could be used such that the average hop distance of the multicast packet is somewhat decreased. In doing so, the maximum hop distance between a given source node and multicast destination node is approximately $N/2$ as opposed to $(N-1)$ if only one direction is used for sending multicast copies. Building on the analysis presented in this article, this modification with transmissions in both directions is studied in [34].

APPENDIX A
CALCULATION OF $E[H_{\Lambda,\lambda}]$ FOR UNI-DIRECTIONAL RING

Inserting (53) and (54) in (59) we obtain

$$E[H_{\Lambda,\lambda}] = \lambda(1-P(F_{\Lambda,\lambda}=0)) + \Lambda(\eta-1) - \Lambda(\eta-1)P(F_{\Lambda,\lambda}=0) - \Lambda\sum_{l=1}^{N-1}\mu_l\sum_{k=0}^{\eta-2}\frac{1}{\binom{N-1}{l}}\sum_{i=1}^{k+1}\binom{k+1}{i}\binom{N-\eta-1}{l-i} \tag{130}$$

$$= (\lambda+\Lambda(\eta-1))(1-P(F_{\Lambda,\lambda}=0)) - \Lambda\sum_{l=1}^{N-1}\mu_l\sum_{k=0}^{\eta-2}\frac{1}{\binom{N-1}{l}}\left(\binom{N+k-\eta}{l}-\binom{N-\eta-1}{l}\right) \tag{131}$$

$$= (\lambda+\Lambda(\eta-1))(1-P(F_{\Lambda,\lambda}=0)) - \Lambda\sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}} \times\left(\binom{N-1}{l+1}-\binom{N-\eta}{l+1}-(\eta-1)\binom{N-\eta-1}{l}\right) \tag{132}$$

$$= \lambda(1-P(F_{\Lambda,\lambda}=0)) + \Lambda(\eta-1) - \Lambda N\sum_{l=1}^{N-1}\frac{\mu_l}{l+1} + \Lambda\sum_{l=1}^{N-1}\mu_l\frac{\binom{N-\eta}{l+1}}{\binom{N-1}{l}} + \Lambda \tag{133}$$

which results in (60).

APPENDIX B
CALCULATION OF $E[H_{\Lambda,\lambda}]$ FOR BI-DIRECTIONAL RING

Inserting (73), (75), (77), and (79) in (70), we obtain

$$E[H_{\Lambda,\lambda}|F_{\Lambda,\lambda}=i] = \sum_{k=0}^{\lfloor\frac{\eta}{2}-\frac{\lambda}{\Lambda}\rfloor}(\lambda+k\Lambda)\frac{i}{\eta}\frac{\binom{k}{i-1}}{\binom{\eta-1}{i-1}} + (N-\lambda-k\Lambda)\frac{i}{\eta}\frac{\binom{\eta-k-1}{i-1}}{\binom{\eta-1}{i-1}} \cdot\left(1-\frac{\binom{\eta-2k-1}{i-1}}{\binom{\eta-k-1}{i-1}}\right) + \sum_{k=\lfloor\frac{\eta}{2}-\frac{\lambda}{\Lambda}\rfloor+1}^{\eta-1}(\lambda+k\Lambda)\frac{i}{\eta}\frac{\binom{k}{i-1}}{\binom{\eta-1}{i-1}} \cdot\left(1-\frac{\binom{2k-\eta}{i-1}}{\binom{k}{i-1}}\right) + (N-\lambda-k\Lambda)\frac{i}{\eta}\frac{\binom{\eta-k-1}{i-1}}{\binom{\eta-1}{i-1}}. \tag{134}$$

Denoting the terms associated with $\frac{\binom{\eta-2k-1}{i-1}}{\binom{\eta-k-1}{i-1}}$ and $\frac{\binom{2k-\eta}{i-1}}{\binom{k}{i-1}}$ by R, we obtain

$$E[H_{\Lambda,\lambda}|F_{\Lambda,\lambda}=i] = \frac{i}{\eta}\sum_{k=0}^{\eta-1}\left\{(\lambda+k\Lambda)\frac{\binom{k}{i-1}}{\binom{\eta-1}{i-1}} + (N-\lambda-k\Lambda)\frac{\binom{\eta-k-1}{i-1}}{\binom{\eta-1}{i-1}}\right\} - R \tag{135}$$

$$= \frac{i}{\eta}\left\{\sum_{k=0}^{\eta-1}(\lambda+k\Lambda)\frac{\binom{k}{i-1}}{\binom{\eta-1}{i-1}} + \sum_{l=0}^{\eta-1}(-\lambda+(l+1)\Lambda)\frac{\binom{l}{i-1}}{\binom{\eta-1}{i-1}}\right\} - R \tag{136}$$

$$= \frac{i}{\eta}\sum_{k=0}^{\eta-1}(2k+1)\Lambda\frac{\binom{k}{i-1}}{\binom{\eta-1}{i-1}} - R \tag{137}$$

where in (136) we denote $l=\eta-1-k$. Noting that $\lfloor\eta/2-\lambda/\Lambda\rfloor = \lfloor(\eta-1)/2\rfloor$ we obtain for R

$$R = \frac{i}{\eta\binom{\eta-1}{i-1}}\left\{\sum_{k=0}^{\lfloor\frac{\eta-1}{2}\rfloor}(\eta\Lambda-\lambda-k\Lambda)\binom{\eta-2k-1}{i-1} + \sum_{k=\lfloor\frac{\eta-1}{2}\rfloor+1}^{\eta-1}(\lambda+k\Lambda)\binom{2k-\eta}{i-1}\right\} \tag{138}$$

$$= \frac{1}{\binom{\eta}{i}}\left\{\lambda\sum_{j=1}^{\eta}\binom{\eta-j}{i-1}(-1)^j + \sum_{k=0}^{\lfloor\frac{\eta-1}{2}\rfloor}(\eta-k)\Lambda\binom{\eta-2k-1}{i-1} + \sum_{k=\lfloor\frac{\eta-1}{2}\rfloor+1}^{\eta-1}k\Lambda\binom{2k-\eta}{i-1}\right\}. \tag{139}$$

Inserting (137) and (139) in (69) we obtain for $\lambda = 1,\ldots,\Lambda-1$

$$E[H_{\Lambda,\lambda}] = \sum_{i=1}^{\eta}\sum_{l=1}^{N-1}\mu_l\frac{\binom{N-1-\eta}{l-i}}{\binom{N-1}{l}} \cdot\left\{-\lambda\sum_{j=1}^{\eta}\binom{\eta-j}{i-1}(-1)^j + \Lambda\left(\binom{\eta}{i}\left(2i\frac{\eta+1}{i+1}-1\right) - \sum_{k=0}^{\lfloor\frac{\eta-1}{2}\rfloor}(\eta-k)\binom{\eta-2k-1}{i-1} - \sum_{\lfloor\frac{\eta-1}{2}\rfloor+1}^{\eta-1}k\binom{2k-\eta}{i-1}\right)\right\} \tag{140}$$

$$
\begin{aligned}
&= \sum_{l=1}^{N-1} \frac{-\lambda\mu_l}{\binom{N-1}{l}} \sum_{j=1}^{\eta}(-1)^j \binom{N-1-j}{l-1} \\
&+ \Lambda \sum_{l=1}^{N-1} \frac{\mu_l}{\binom{N-1}{l}} \sum_{i=1}^{\eta} \binom{N-1-\eta}{l-i} \\
&\times \left(2i\binom{\eta+1}{i+1} - \binom{\eta}{i}\right) \\
&- \Lambda \sum_{l=1}^{N-1} \frac{\mu_l}{\binom{N-1}{l}} \sum_{k=0}^{\lfloor\frac{\eta-1}{2}\rfloor} (\eta-k) \\
&\times \sum_{i=1}^{\eta} \binom{N-1-\eta}{l-i}\binom{\eta-2k-1}{i-1} \\
&- \Lambda \sum_{l=1}^{N-1} \frac{\mu_l}{\binom{N-1}{l}} \\
&\times \sum_{k=\lfloor\frac{\eta-1}{2}\rfloor+1}^{\eta-1} k \sum_{i=1}^{\eta} \binom{N-1-\eta}{l-i}\binom{2k-\eta}{i-1}
\end{aligned} \tag{141}
$$

where the term $\binom{N-1-j}{l-1}$ in the first line of (141) was obtained by noting that the hypergeometric distribution sums to one. Similarly, we obtain

$$
\sum_{i=1}^{\eta} \binom{N-1-\eta}{l-i}\binom{\eta-2k-1}{i-1} = \binom{N-2k-2}{l-1} \tag{142}
$$

$$
\sum_{i=1}^{\eta} \binom{N-1-\eta}{l-i}\binom{2k-\eta}{i-1} = \binom{N+2k-2\eta-1}{l-1}. \tag{143}
$$

Furthermore, we can simplify the second line of (141) to (144) and (145) shown at the bottom of the page.

Hence

$$
\begin{aligned}
E[H_{\Lambda,\lambda}] = &-\lambda \sum_{l=1}^{N-1} \frac{\mu_l}{\binom{N-1}{l}} \sum_{j=1}^{\eta}(-1)^j \binom{N-1-j}{l-1} \\
&+ \Lambda \sum_{l=1}^{N-1} \frac{\mu_l}{\binom{N-1}{l}} \left\{ (2\eta+1)\binom{N-1}{l} \right. \\
&+ \binom{N-1-\eta}{l} + 2\binom{N-1-\eta}{l+1} \\
&-2\binom{N}{l+1} - \sum_{k=0}^{\lfloor\frac{\eta-1}{2}\rfloor} (\eta-k)\binom{N-2k-2}{l-1} \\
&\left. - \sum_{k=\lfloor\frac{\eta-1}{2}\rfloor+1}^{\eta-1} k \binom{N+2k-2\eta-1}{l-1} \right\}
\end{aligned} \tag{146}
$$

which simplifies to (80).

Appendix C
Calculation of $E[H_{\Lambda,\Lambda}]$ for Bi-Directional Ring

Exploiting again the property of the hypergeometric distribution to sum to one, we obtain from (85)

$$
\begin{aligned}
E[H_{\Lambda,\Lambda}] = \Lambda &\sum_{l=1}^{N-1} \frac{\mu_l}{\binom{N-1}{l}} \\
&\cdot \left\{ 2\sum_{k=1}^{\eta-1} k \binom{N-\eta+k-1}{l-1} \right. \\
&- \sum_{k=1}^{\lfloor\eta/2\rfloor} (\eta-k)\binom{N-2k}{l-1} \\
&\left. - \sum_{k=\lfloor\eta/2\rfloor+1}^{\eta-1} k\binom{N+2k-2\eta-1}{l-1} \right\}.
\end{aligned} \tag{147}
$$

This can be reduced further as

$$
\begin{aligned}
&2\sum_{k=1}^{\eta-1} k\binom{N-\eta+k-1}{l-1} \\
&= 2\left\{ \sum_{k=1}^{\eta-1}(N-\eta+k)\binom{N-\eta+k-1}{l-1} \right. \\
&\qquad \left. - \sum_{k=1}^{\eta-1}(N-\eta)\binom{N-\eta+k-1}{l-1} \right\}
\end{aligned} \tag{148}
$$

$$
\begin{aligned}
&= 2l\sum_{k=1}^{\eta-1}\binom{N-\eta+k}{l} \\
&\quad - 2(N-\eta)\sum_{k=1}^{\eta-1}\binom{N-\eta+k-1}{l-1}
\end{aligned} \tag{149}
$$

$$
\Lambda \sum_{l=1}^{N-1} \frac{\mu_l}{\binom{N-1}{l}} \sum_{i=1}^{\eta} \left\{ 2(i+1)\frac{(\eta+1)!}{(i+1)!(\eta-i)!} - 2\binom{\eta+1}{i+1} - \binom{\eta}{i} \right\} \binom{N-1-\eta}{l-i} \tag{144}
$$

$$
= \Lambda \sum_{l=1}^{N-1} \frac{\mu_l}{\binom{N-1}{l}} \left\{ (2\eta+1)\binom{N-1}{l} + \binom{N-1-\eta}{l} + 2\binom{N-1-\eta}{l+1} - 2\binom{N}{l+1} \right\}. \tag{145}
$$

$$= 2l\left\{\binom{N}{l+1} - \binom{N-\eta+1}{l+1}\right\} - 2(N-\eta)\left\{\binom{N-1}{l} - \binom{N-\eta}{l}\right\}. \quad (150)$$

Inserting (150) into (147) and denoting the last two sums over k in (147) by R, we obtain

$$E[H_{\Lambda,\Lambda}] = \Lambda\sum_{l=1}^{N-1}\mu_l\left\{2l\frac{N}{l+1} - 2l\frac{\binom{N-\eta+1}{l+1}}{\binom{N-1}{l}} + 2(N-\eta)\frac{\binom{N-\eta}{l}}{\binom{N-1}{l}}\right\} - 2(N-\eta)\Lambda - \Lambda\sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}}R \quad (151)$$

$$= 2N\Lambda - 2\Lambda N\sum_{l=1}^{N-1}\frac{\mu_l}{l+1} - 2\Lambda\sum_{l=1}^{N-1}l\mu_l\frac{\binom{N-\eta+1}{l+1}}{\binom{N-1}{l}} + 2\Lambda(N-\eta)\sum_{l=1}^{N-1}\mu_l\frac{\binom{N-\eta}{l}}{\binom{N-1}{l}} - 2(N-\eta)\Lambda - \Lambda\sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}}R \quad (152)$$

and

$$E[H_{\Lambda,\Lambda}] = 2N - 2\Lambda N\sum_{l=1}^{N-1}\frac{\mu_l}{l+1} + 2\Lambda\sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}} \cdot\left\{(N-\eta)\binom{N-\eta}{l} - l\binom{N-\eta+1}{l+1}\right\} - \Lambda\sum_{l=1}^{N-1}\frac{\mu_l}{\binom{N-1}{l}}R \quad (153)$$

which further simplifies to (86).

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