Theory and Applications of Gaussian Quadrature Methods

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ABSTRACT

Gaussian quadrature is a powerful technique for numerical integration that falls under the broad category of spectral methods. The purpose of this work is to provide an introduction to the theory and practice of Gaussian quadrature. We study the approximation theory of trigonometric and orthogonal polynomials and related functions and examine the analytical framework of Gaussian quadrature. We discuss Gaussian quadrature for bandlimited functions, a topic inspired by some recent developments in the analysis of prolate spheroidal wave functions. Algorithms for the computation of the quadrature nodes and weights are described. Several applications of Gaussian quadrature are given, ranging from the evaluation of special functions to pseudospectral methods for solving differential equations. Software realization of select algorithms is provided.

KEYWORDS

numerical integration, Gaussian quadrature, approximation theory, trigonometric and orthogonal polynomials, bandlimited functions, prolate spheroidal wave functions, spectral methods

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